BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

INCORPORATION OF THE AMERICAN MATHEMATICAL SOCIETY

One of the milestones in the progress of the Society was passed on October 27, 1923, when the unincorporated body turned over the conduct of its affairs to the corporation known as the American Mathematical Society. An announcement was made in this BULLETIN for June, 1923, that a certificate of incorporation had been filed in the District of Columbia and that a meeting of the corporation had been held; but it was later ascertained that this meeting on May 4, 1923, was legally irregular. In consequence, O. S. Adams, Harry English, J. T. Erwin and H. L. Hodgkins met in the office of Dean Hodgkins, 2033 G Street, N. W., Washington, D. C., on October 22, 1923, at 4.00 p. m., for the purpose of arranging all preliminaries necessary to a final transfer of authority to the corporation at the October meeting of the Society. President Veblen was present by invitation.

Dean Hodgkins was elected President of the incorporators and Professor Erwin, Secretary. A waiver of the notice of meeting, signed by all five incorporators, was read.

A set of By-Laws drawn up by a committee of the Society was adopted. According to the revised By-Laws, Ex-Presidents of the Society will after 1923 no longer be ex-officio members of the Council. The designation "Chicago Section" has been dropped, since the meetings in that region have for many years been meetings of the Society. With the addition of a third Vice-President, it is expected that

in the future such meetings will be presided over by the President or a Vice-President. To provide secretaries for both regions in which regular meetings of the Society are held, the office of Assistant Secretary has been created in addition to that of Secretary. With these exceptions, the revised By-Laws in general conform as closely to the By-Laws of the unincorporated body as the legal requirements permit. The only essential change *necessitated by incorporation* is the addition of a Board of Trustees to take charge of the financial affairs of the Society. The Council will as heretofore have entire charge of its scientific activities. The By-Laws will be published in full in the List of Members to be issued in the autumn of 1924.

All members of the unincorporated body were elected to membership in the corporation. To fulfil the requirements of the Certificate of Incorporation, the following Board of Trustees was appointed by the incorporators, the list of thirty-one being made up with a view to ensuring the necessary quorum of a majority at the session on October 27 and at the Annual Meeting for 1923: J. W. Alexander, R. C. Archibald, B. A. Bernstein, G. D. Birkhoff, E. W. Brown, F. N. Cole, L. P. Eisenhart, H. B. Fine, W. B. Fite, T. C. Fry, H. E. Hawkes, R. Henderson, H. L. Hodgkins, E. V. Huntington, S. A. Joffe, O. D. Kellogg, E. H. Moore, W. F. Osgood, Anna J. Pell, M. I. Pupin, R. G. D. Richardson, J. F. Ritt, L. P. Siceloff, Clara E. Smith, D. E. Smith, W. M. Strong, H. W. Tyler, Oswald Veblen, H. S. White, J. K. Whittemore, J. W. Young.

The incorporators authorized the Trustees to take over the assets of the unincorporated body and to assume its liabilities. So far as was consistent with the By-Laws, they elected as officers of the corporation the officers of the old organization and for the same terms. The members of the Editorial Committee for the Transactions, formerly members of the Council but not officers, were elected officers with the same terms; as their terms expire, their successors will be elected by the Society, instead of being appointed by

the Council. The name of the Committee of Publication was changed to Editorial Committee for the Bulletin, the term of office being adjusted so that instead of all members retiring in 1923, E. R. Hedrick continues until December, 1924, and J. W. Young until December, 1925. The appointed committees of the new organization correspond precisely to those of the old.

As a further necessary step in the proceedings, a meeting of the Board of Trustees was held in the Trustees Room of Columbia University on October 27, 1923, at 9.30 a. m. The call and the waiver, bearing the signatures of all the trustees, were read. Twenty-eight of the trustees responded to the roll-call. President Veblen was elected Chairman of the Board and R. G. D. Richardson, Secretary. An executive committee was appointed, consisting of Oswald Veblen, H. E. Hawkes, Robert Henderson, R. G. D. Richardson, and H. W. Tyler. Resolutions were passed authorizing the taking over of the assets and liabilities of the unincorporated Society and the general dispatch of necessary business.

The final stage in the proceedings was reached at the morning sessions with the adoption of resolutions by the Council and Society transferring all property and incumbrances to the Corporation.

The By-Laws of the incorporated body specify that after May 3, 1924, the Board of Trustees shall consist of five members, and nominations by the Council for these trustees were ordered printed on the ballot to be voted on at the Annual Meeting of 1923.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SOCIETY

The two hundred thirty-first regular meeting of the Society was held at Columbia University on Saturday, October 27, 1923, extending through the usual morning and afternoon sessions. This meeting marks the transition of the Society into an incorporated body.

The Society was called to order for the transaction of business at 11.15 a. m., in Room 305 of Schermerhorn Hall, Columbia University, New York City, some fifty-five members being present. The Secretary recalled the action of the Society in previously endorsing incorporation and reported the formal action of the Council recommending the final steps. President Veblen made a general statement concerning the procedure adopted on legal advice, and recounted briefly the stages of the incorporation process, which are reported in another article in this issue.

The following resolution was then adopted unanimously:

WHEREAS, a corporation has been organized under the laws of the District of Columbia, known as the American Mathematical Society, for the purpose of carrying on the work heretofore conducted by this Society; and

WHEREAS, it is the judgment of this Society that for the purpose of more efficiently prosecuting said work it is expedient to transfer to the said new corporation the entire property and assets of this Society;

NOW, THEREFORE, BE IT RESOLVED, that the officers of this Society be and they are hereby authorized and directed to transfer to the American Mathematical Society, a corporation as aforesaid, all the property and assets of this Society, including its bank deposits and choses in action, provided the said corporation shall assume and agree to pay all indebtedness and financial obligations of every kind and description for which this Society may now be liable;

AND BE IT FURTHER RESOLVED, that the proper officers

of this Society be and they are hereby empowered, to execute, acknowledge, and deliver, all contracts, deeds, and other documents necessary and proper for the carrying into effect this resolution.

The attendance at the scientific Sessions included the following sixty-eight members of the Society:

Alexander, Archibald, Ballantine, Bernstein, Birkhoff, E. W. Brown, Abraham Cohen, Cole, Cowley, L. D. Cummings, Douglas, Edmondson, Eisenhart, Fine, Fiske, Fry, Ginsburg, Gleason, Glenn, Grove, Hausle, Hawkes, Hazlett, Hebbert, Robert Henderson, Hille, Himwich, Hodgkins, Hotelling, Huntington, Dunham Jackson, Joffe, Kasner, Kellogg, Kircher, Kline, Langman, MacColl, MacNeish, Meder, Mirick, Molina, Northcott, Osgood, Pell, Pfeiffer, Post, Rainich, Ranum, Reddick, R. G. D. Richardson, Ritt, Schub, Seely, Siceloff, Simons, C. E. Smith, D. E. Smith, Sosnow, Strong, Tyler, Veblen, H. E. Webb, Weiss, M. E. Wells, H. S. White, Whittemore, J. W. Young.

At the meeting of the Council, the following forty-nine persons were elected to membership in the Society:

Miss Ethel Louise Anderton, New Haven, Conn.;
Professor Victor-Elezéar Beaupré, University of Montreal;
Dr. Harry Albert Bender, University of Illinois;
Professor Jacob Roy Bender, University of Idaho;
Miss Lucy Elizabeth Berger, College of William and Mary;
Professor Harry Birchenough, State College for Teachers, Albany, N. Y.;
Mr. Harley W. Chandler, University of Florida;
Professor Myrtie Collier, Southern Branch of the University of California;
Mr. Arthur Herbert Copeland, Harvard University;
Professor Charles Edgar Corbin, College of the Pacific;
Mr. Frederic Warren Darling, United States Coast and Geodetic Survey;
Mr. Charles Herbert Davis, Yale Observatory;
Mr. Joseph Oscar Eckersley, Department of Plant and Structures, New York City;
Mr. Jekuthiel Ginsburg, New York City;
Mr. Kut William Halbert, Harvard University;
Dr. Victor August Hoersch, University of Illinois;
Mr. Julian Laurence Holley, Harvard University;
Dean Charles Ellsworth Horne, College of Agriculture and Mechanic Arts, Mayaguez, Porto Rico;
Professor Charles Applewhite Isaacs, State College of Washington;
Dr. Hamilton M. Jeffers, University of Iowa;
Professor William H. Kirchner, University of Minnesota;
Professor Alois Francis Kovarik, Yale University;
Mr. Harry Isler Lane, University of South Dakota;
President William Orville Mendenhall, Friends University;
Mr. Aristotle Michal, Rice Institute;
Miss Helen Moon, Lincoln National Life Insurance Company, Fort Wayne, Ind.;
Miss Thirza Adaline Mossman, Kansas State Agricultural College;
Professor George Alexander Newton, Trinity University;
Mr. Frank C. Ogg, University of Illinois;

Professor James Robert Overman, State Normal College, Bowling Green, Ohio;
Professor Arthur Pelletier, Ecole Polytechnique of Montreal;
Mr. Frederick William Perkins, Harvard University;
Professor Edward C. Phillips, Woodstock College;
Mr. Joseph Crawford Polley, Colgate University;
Mr. George Yuri Rainich, Johns Hopkins University;
Professor Charles Henry Rawlins, United States Naval Academy;
Mr. Perry Dean Schwartz, New Haven, Conn.;
Professor Isaac F. Seiverling, State Normal School, Millersville, Pa.;
Mr. Robert Vincent Sinnott, Hartford, Conn.;
Mr. Morris Miller Slotnick, Harvard University;
Mr. Charles Calvin Steck, Tonawanda, N. Y.;
Dr Tracy Yerkes Thomas, University of Chicago;
Mr. Kamihayashi Tokusaburo, Iwateken, Japan;
Professor Richard Chace Tolman, California Institute of Technology;
Mr. Glen M. V. Tryon, Fenton, Mich.;
Mr. George Peterkin Unseld, Technical High School, Salt Lake City;
Mr. John Russell Vatnsdal, Yale University;
Mr. Charles Weiss, New York City;
Professor Joseph Wilczewski, St. Xavier College, Cincinnati.

Fourteen applications for membership in the Society were received.

The Committee on Printing announced that arrangements were being made to print both the BULLETIN and the TRANSACTIONS in Hamburg, Germany, during the coming year.

Announcement was made of the appointment of Professors H. E. Hawkes (chairman), E. W. Brown, J. L. Coolidge, and H. S. White as the Committee on the first Josiah Willard Gibbs lecture; of Professor E. E. Moots as representative at the inauguration of President Updegraff of Cornell College on October 19; of Professor E. V. Huntington as representative at the inauguration of President Comstock of Radcliffe College on October 20; and of Professor W. H. Roever, as representative at the inauguration of President Hadley of Washington University on November 10.

Votes of thanks were extended to Boston College for a gift toward the Endowment Fund; to Professor W. H. Roever for a gift toward the expenses of the membership campaign; to the Committee on Incorporation and the incorporators; and to Messrs. Clephane, Harriman, Latimer, and Putnam for legal advice, tendered without fee, in connection with the incorporation proceedings.

It was decided to hold a meeting of the Council in

conjunction with the western meeting of the Society at Chicago in April, 1924.

A list of nominations for trustees, officers, and other members of the Council was adopted unanimously and ordered printed on the ballot for the annual election.

President Veblen presided at the sessions of the Society, relieved in the morning by Vice-President H. W. Tyler and in the afternoon by Ex-President E. W. Brown. The first part of the afternoon session was devoted to a paper read, at the request of the Programme Committee, by Professor Anna J. Pell, on *Bilinear and quadratic forms in infinitely many variables.* Titles and abstracts of the other papers read at this meeting follow below. Professor Pólya was introduced by Professor Birkhoff. Dr. Douglas's second paper, and the papers of Professor Lipka, Professor Pólya, Dr. Murray, and Professors Alexander and Richardson were read by title.

1. Professor L. P. Eisenhart: *Spaces of continuous matter in general relativity.*

If the field equations for a perfect fluid are taken as

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -k(\sigma u_i u_j - p g_{ij})$$

where R_{ij} is the contracted Riemann tensor with respect to the fundamental tensor g_{ij}, R is the scalar curvature, and σ and p are scalars, the determinant equation $|R_{ij} + \varrho g_{ij}| = 0$ has a simple root and a triple root with simple elementary divisors. Conversely, when the roots of this equation for a space satisfy these conditions and at each point the fundamental quadratic form is reducible to $-dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$ at each point, the space may be interpreted as the space-time continuum of a perfect fluid. If ϱ' is the simple root and ϱ'' the triple root, then $\sigma = (\varrho' - \varrho'')/k$ and $p = (\varrho' + \varrho'')/2k$. The contravariant components u^i of the line of flow are given by

$$(R_{ij} + \varrho' g_{ij}) u^i = 0.$$

2. Professor J. K. Whittemore: *The deformation of ruled surfaces.*

In this paper it is shown that the equations of any ruled surface, rulings u constant, whose linear element is

given by $ds^2 = [(v-\alpha)^2 + \beta^2]\,du^2 + dv^2$, where α and β are arbitrary functions of u, are $x = Av + \int(P\beta - L\alpha)\,du$ with similar equations for y and z, where A, L and P are the cosines of the angles of the x-axis and the tangent, principal normal, and binormal, respectively, of a curve of unit curvature and arbitrary torsion, and u is the arc of this curve. A property of the surface depending on the torsion of this curve is given. The method consists in determining from the Codazzi equations the single unknown coefficient of the second fundamental form of the required surface, then integrating the partial differential equations for the direction cosines of the normal and the tangents to the parametric curves.

3. Mr. G. Y. Rainich: *Analytic vector functions.*

This theory is analogous to the theory of analytic functions of a complex variable; the place of the complex number is occupied by the three-dimensional vector and the part of the product is played by a bilinear vector function, the basis. An analytic function is defined with the aid of a power series; the region of convergence is a parallelepiped or an elliptic cylinder; the condition of analyticity is given by a set of two differential vector equations (analogy with the Cauchy-Riemann equations) or by the vanishing of a contour integral (analogy with the Cauchy-Morera theorem). These functions have no independent importance, because all functions analytic on the same basis have one or three directions along which they have particular properties. To get rid of this peculiarity we consider analytic functions of two variables; substituting for each of these variables a linear vector function of the same variable z we get a function of z which we call semi-analytic; the functions of a semi-analytic class are characterized by one differential vector equation or by the vanishing of a surface integral; they can be expanded into double series. Analogous functions can be formed also starting from ordinary analytic functions.

4. Dr. Jesse Douglas: *Systems of ∞^{2n-2} curves in a Riemann space in which the sum of the angles of every triangle formed by three of the curves is two right angles.*

This paper proves the following theorem. If in a Riemann space of three or more dimensions, $ds^2 = \Sigma a_{ij}\,dx^i\,dx^j$, there exists a system of ∞^{2n-2} curves, defined by a set of differential equations of the form $d^2x^i/ds^2 = F_i(x, dx/ds)$, which has the

property that in every triangle formed by three of the curves the sum of the angles is equal to π, then the Riemann space must be conformally equivalent to a euclidean space, and in such a way that the ∞^{2n-2} curves correspond to the straight lines of the euclidean space.

5. Dr. Jesse Douglas: *Necessary and sufficient conditions that a system of ∞^4 curves in space consist of the mutual intersections of ∞^3 surfaces.*

The ∞^4 straight lines of space are the mutual intersections of the ∞^3 planes. Not every system of ∞^4 curves in space has a so related system of ∞^3 surfaces. Assuming the curve system to be defined by the differential equations $y'' = F(x,y,z,y',z'), z'' = G(x,y,z,y',z')$ the present paper derives necessary and sufficient conditions in the form of three partial differential equations of the second order bearing on F and G.

6. Professor Joseph Lipka: *On Ricci's coefficients of rotation.*

Ricci's coefficients of rotation, γ_{ijk}, play an important rôle in the absolute calculus of any space whatsoever. He first defines the γ's analytically and then shows their geometric meaning be means of the tangent spaces. Following a suggestion made to the writer by Professor Levi-Civita, the γ's are here defined geometrically by means of the notion of parallelism. As a consequence, the expressions for the geodesic curvature and several other invariants in terms of the γ's are all found and interpreted intrinsically. Applying the notion of conformal parallelism, the γ's are generalized so that they involve an arbitrary function of the coordinates, and geometric interpretations of certain invariants in the generalized coefficients are given.

7. Professor Joseph Lipka: *Types of alignment charts in three variables.*

This paper contains a study of a systematic classification of alignment or nomographic charts, based upon parallel coordinates. In particular, it gives the specific form which an equation in three variables, $f(u, v, w) = 0$, must have in order that it may be represented by certain combinations of curved and straight line scales. It also gives the equations of the scales in each case, and their method of construction.

8. Professor Georg Pólya: *On the mean-value theorem corresponding to a given linear homogeneous differential equation.*

This paper, which will appear in the TRANSACTIONS OF THIS SOCIETY, deals with the conditions which must be imposed on a linear differential form $L(y)$ of the nth order in order that whenever y vanishes $n+1$ times on the given interval, $L(y)$ shall necessarily vanish on that interval at least once.

9. Dr. F. H. Murray: *Note on stability à la Poisson.*

This paper appears in full in the present issue of this BULLETIN.

10. Professor J. W. Alexander: *On infinitely connected plane regions.*

Some theorems are proved on the topology of infinitely connected plane regions, and a class of regions is determined such that each region is characterized by a pair of invariants, one finite, the other transfinite.

11. Professor J. W. Alexander: *On the deformation of an n-cell.*

Let $r' = R(r, \theta)$, $\theta' = \Theta(r, \theta)$ be the equations of a one-to-one continuous transformation T carrying the interior and boundary of the unit circle into itself and leaving all points on the boundary of the unit circle invariant. The transformation may be extended to the entire plane by saying that it leaves all points without the unit circle invariant. Construct, now, a transformation T_λ such that $r' = \lambda R(r/\lambda, \theta)$, $\theta' = \Theta(r/\lambda, \theta)$ for $\lambda > 0$ and such that T_0 is the identity. Then, as λ varies from 0 to 1, T_λ determines a one-to-one continuous deformation generating the transformation T and leaving the boundary of the unit circle pointwise invariant. The theorem generalizes to n dimensions.

12. Professor R. G. D. Richardson: *On the reality of the zeros of a λ determinant.*

This paper appeared in full in the December number of this BULLETIN.

13. Professor E. V. Huntington: *Sets of completely independent postulates for cyclic order.*

The author gives two sets of five postulates for the theory

of cyclic order, each set being shown to be "completely independent" in the sense of E. H. Moore. An abstract of some of the results appeared in the PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 2 (1916), pp. 630–631. The paper is closely related to the exhaustive treatment of *Sets of independent postulates for betweenness*, published by E. V. Huntington and J. R. Kline in the TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), pp. 301–325, though the results in the present case are much more easily obtained. The paper will be offered to the TRANSACTIONS.

14. Professor Dunham Jackson: *Some corollaries of Bernstein's theorem.*

The theorem of Bernstein referred to in the title states that if $T_n(x)$ is a trigonometric sum of order n, and if $|T_n(x)| \leq L$ for all values of x, then $|T_n'(x)|$ can never exceed nL. The hypothesis requires that the condition on $|T_n(x)|$ be satisfied for *all* (real) values of x, or, what amounts to the same thing, that it be satisfied throughout an interval of length 2π. The present paper obtains results of similar character, though less simple in form, on the assumption merely that the condition is satisfied throughout an interval of length less than 2π, the conclusions then being valid for the restricted interval also. These results are deduced indirectly from Bernstein's theorem itself.

15. Dr. E. L. Post: *Theory of generalized differentiation.*

The theory of derivatives of non-integral orders, begun by Liouville and Riemann, may be said to be complete. On the other hand, the theory of more general operators considered as functions of D, the derivative, is fragmentary. In the present paper, the problem is attacked by an entirely new method, and results are obtained that generalize those of Riemann. Although the present investigation was inspired purely by an interest in the bizarre, it has been found to connect with many fields in the most recent branches of analysis. In particular, it has applications in the theory of the Laplace transformation, Volterra's permutable functions and functions of composition (of the closed cycle group), and the Heaviside operational classes.

R. G. D. RICHARDSON,
Secretary.

THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION

The forty-second regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of Southern California, Los Angeles, on September 18, 1923. Professor Cajori, chairman of the section, opened the meeting and presided until the election of the new chairman. The attendance included the following seventeen members of the Society:

Bateman, Cajori, A. F. Carpenter, Daus, Glazier, Hoskins, Glenn James, F. R. Morris, Russell, Sherwood, Showman, Steed, Touton, Wear, Willett, Winger, Wolfe.

The election of officers for the ensuing year resulted as follows: Chairman, Professor A. F. Carpenter; Secretary-Treasurer, Professor B. A. Bernstein; Program Committee, Professors B. A. Bernstein, D. N. Lehmer, E. T. Bell.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professor Bell, Griffin, and Porter, and Mr. Romig were read by title. Mr. Romig was introduced by Professor Cajori.

1. Professor E. T. Bell: *Notes on recurring series of the third order.*

Important new historical material regarding the u, v of Lucas is given in the first part of this paper. The remainder is devoted to the algebra of the three linearly independent functions defined by a recurrence relation of the third order. The most general possible addition theorems for each of the functions are obtained. By slight changes in notation all results of the paper can be extended at once to series of order $n > 3$. As was to be expected, the results are much more complicated than those for Lucas' functions. There are a few applications to Diophantine analysis.

2. Professor E. T. Bell: *Analogies between elliptic functions and the u, v of Lucas.*

This paper appeared in full in the November, 1923, number of this BULLETIN.

3. Professor Florian Cajori: *Did Pitiscus use the decimal point?*

While Pitiscus is his *Trigonometry* of 1612 used fractions extensively, he did not use the point as separatrix between units and tenths. Most modern historians of mathematics state incorrectly that he used the decimal point in his trigonometric tables of 1612. The point does occur hundreds of times, but simply as a mark to divide the numeral figures into groups easier to read and copy.

4. Professor Florian Cajori: *Empirical generalizations on the growth of mathematical notations.*

Several empirical observations on the nature and growth of notation are given, which point to haphazard developments usually failing of general adoption, and which set forth the difficulties encountered in securing universal notations. The efforts of international committees towards uniformity of notation are recounted.

5. Professor A. F. Carpenter: *Linear complexes associated with the general ruled surface.*

In a paper read before the San Francisco Section of the Society in April, 1923, a number of theorems were proved concerning the primary and secondary flecnodal and complex cubics associated with each element of a ruled surface. Each of these cubics determines a linear complex. In the present paper, the author discusses the properties of these complexes and certain new cubics defined by means of them.

6. Professor A. F. Carpenter: *Note on a net of curves on a general surface.*

By a method analogous to that which enables us to define lines of curvature of a surface, and the Dupin indicatrix at a point, it is possible to obtain a quadratic differential form whose vanishing determines the extreme values of the geodesic torsion, and which may therefore be taken as defining a net of curves on the surface. These curves, which may properly be called lines of torsion, constitute an orthogonal system, the two curves of this system through any surface point bisecting the angles between the lines of curvature. The maximum and minimum values of the geodesic torsion are equal but opposite in sign for all

real surfaces, and the indicatrix for the geodesic torsion is a rectangular hyperbola whose asymptotes coincide with the tangents to the lines of curvature.

7. Dr. P. H. Daus: *Numerical illustrations of a theorem in ternary continued fractions.*

Connected with the equation $x^3+qx-(p_1^3+p_1q+1)=0$, there exists a normal ternary continued fraction expansion $(p_1, p_1^3+q;\ 2p_1, 3p_1^2+q;\ 3p_1, 3p_1^2+q;)$. This note considers certain numerical cases in the light of a theorem by Lehmer (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 4 (1918), p. 360) which determines whether the ratios $A_n:B_n:C_n$ formed from the nth convergent set do or do not approach limits.

8. Mr. H. P. Robertson: *The absolute differential calculus of a non-Pythagorean, non-Riemannian space.*

Fundamental in the theory of relativity is the assumption that the line element of four-dimensional space-time is represented by the square root of an invariant quadratic differential form, and the subsequent generation of its related tensors by means of the absolute differential calculus. If, instead, it is assumed (a possibility suggested by Riemann) that the line element is represented by the $2m$th root of a differential form of degree $2m$ (m an integer) a new calculus is needed for the generation of tensors connected therewith. The present paper develops such an instrument, which applies to differential forms of degree $2m$ in n variables, and in which the Ricci-Levi-Civita calculus is included as a special case.

9. Dr. Victor Steed: *On a system of equations connected with the lines on the cubic surface.*

To show that there are lines on the cubic surface, Salmon suggested eliminating two of the cartesian coordinates from the equations of a line and the equation of the surface, and then equating to zero the coefficients of the resulting equation in the third coordinate. The four equations in the four coordinates of the line thus obtained are here interpreted as four cubic hypersurfaces in space of four dimensions. If these were not special, they would have, by the theorem of Bezout, 81 distinct points of intersection. The

question of how to account for the fact that these hypersurfaces have only 27 points of intersection in finite space is considered, it being shown that all four hypersurfaces have in common three skew lines lying entirely in the hyperplane at infinity.

10. Professor Harry Bateman: *Derivation of three-dimensional potentials from four-dimensional potentials.*

When a solution of Laplace's equation in four variables is integrated with respect to one of these variables from $-\infty$ to $+\infty$ the result is generally a solution of Laplace's equation in three variables. Since potential functions in four variables are often easier to calculate than analogous potential functions in three variables, the well known device just mentioned may be of real value. At any rate, it leads to some interesting identities, as is shown in this paper.

11. Professor F. L. Griffin: *Curves functionally conjugate with respect to a given curve.*

The author defines a general category of relationships among coplanar curves as follows: Let the radii vectores of three coplanar curves be ϱ_1, ϱ_2, ϱ_3, and let there exist among the radii corresponding to any common value of θ a functional relation $F(\varrho_1, \varrho_2, \varrho_3) = 0$. Regarding one of the curves as a basic reference curve, the other two will be called *functionally conjugate* with respect to the first. Ordinary inverse curves, cissoids of curves, etc., constitute special classes of such conjugates. For various simple selections of the function F and the basic curve, the conjugates of conics are found to be well known curves. This type of relationship admits of generalization to any number of curves and of extension to surfaces and loci in any number of dimensions.

12. Professor M. B. Porter: *Second mean-value theorem for integrals of summable functions.*

This paper appeared in full in the November, 1923, number of this BULLETIN.

13. Mr. H. G. Romig: *Division by zero (a historical study).*

The author describes the first use of zero as a divisor among the Hindus in 628 A. D., and traces historically the

observance of the difference between the infinitesimal and absolute zero, the rigorous exclusion of division by absolute zero by Martin Ohm in 1828 and Axel Harnack in 1881, and the more recent treatment of division by absolute zero in algebra, modern geometry, and function theory.

14. Professor R. M. Winger: *Singular points of self-projective rational septimics.*

In an earlier paper, the author obtained the types of self-projective rational septimics. In the present paper he analyzes the singularities which occur in great variety and considerable complexity. The standard methods of expansion in the neighborhood of the singularity and birational transformation are not suitable for rational curves which are normally given in parametric form. Moreover, the double-point equation is tedious to expand since it leads to an equation of degree 30 in the case of the septimic. Dr. Scott's theory of excess is, however, of great assistance. Two general theorems of interest are proved: (1) If a rational curve of order n is invariant under a cyclic group of order n (and no higher group) this group cannot contain an homology. (2) A rational curve of order n which is invariant under a cyclic G_n (and no higher group) has a multiple point of order greater than 2, having, however, but two distinct parameters. The parameters of the multiple point are the fixed parameters of the binary group, while the multiple point is the only fixed point of the ternary group which lies on the curve.

L. M. Hoskins,
Acting Secretary of the Section.

NOTE ON STABILITY À LA POISSON*

BY F. H. MURRAY

In this note will be developed certain consequences of a theorem due to Poincaré concerning "Stabilité à la Poisson".†

Suppose given a differential system

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2 \cdots x_n), \qquad (i = 1, 2, \ldots, n),$$

such that the functions X_i are analytic in their arguments within an ordinary‡ closed region D, and suppose the equations (1) admit a positive multiplier M within this region. Then

$$(2) \qquad \frac{\partial}{\partial x_1}(MX_1) + \frac{\partial}{\partial x_2}(MX_2) + \cdots + \frac{\partial}{\partial x_n}(MX_n) = 0.$$

In addition one can adopt either of the following hypotheses: (α) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ belonging to D at the time $t = t_0$ remains within or on the boundary of D for all values of t; (β) every solution of (1) which takes on a system of values $(x_1^0, x_2^0, \ldots, x_n^0)$ at $t = t_0$, belonging to an ordinary closed region D' interior to D remains within or on the boundary of D for all values of t.

In the discussion given by Poincaré, hypothesis (α) is made explicitly; but an examination of the argument shows that the same method of reasoning can be applied under hypothesis (β), to the solutions of (1) which, at $t = t_0$, take on values belonging to D'. The result may be stated as follows:

If $P_0(x_1^0, \ldots, x_n^0)$ is any interior point of D', Δ any ordinary closed region containing P_0 and interior to D',

* Presented to the Society, October 27, 1923.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 3, chapter 26.

‡ By "ordinary closed region" we mean an n-dimensional region bounded by an ordinary $(n-1)$-dimensional surface, and possessing a certain volume different from zero.

then there exists at least one point $P'(x_1', x_2', \ldots, x_n')$ belonging to Δ such that, given t_0 and $T > t_0$, there exists a value $\tau > T$ such that the solution of (1) taking on the values $(x_1', \ldots, x_n')$ at $t = t_0$ will take on values $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ belonging to Δ at the time $t = \tau$.

An application will be made to the case in which $X_i(0, 0, \ldots, 0) = 0$, $(i = 1, 2, \ldots, n)$, and in which the system (1) is known to possess *ordinary stability* in a certain neighborhood $|x_i| < A$. Ordinary stability may be said to exist in the given region under the following conditions: given any region (b): $|x_i| \leqq b$, $b \leqq A$, there exists a region (c): $|x_i| \leq c$, $c \leqq b$, such that any solution of (1) which takes on a system of values belonging to (c) at $t = t_0$ lies within (b) for all values of t.

Suppose $c = C$ the value corresponding to $b = A$; the assumption of ordinary stability implies hypothesis (β) in which $D' = (C)$, $D = (A)$. Consequently there is stability à la Poisson within the region (C) if equations (1) admit a positive multiplier M within the region (A). This condition will be satisfied if equations (1) are in the canonical form

$$(3) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad (i = 1, 2, \ldots, m),$$

since equations (2) are satisfied by $M = 1$.

From an examination of the chapter of *Les Méthodes Nouvelles* mentioned above it can easily be seen that the assumptions concerning the functions X_i can be generalized. It is sufficient that the solutions of (1) be continuous functions of the initial values for $t = t_0$, and that they possess continuous partial derivatives of the first order with respect to them. These conditions will be satisfied if the partial derivatives $\partial X_i/\partial x_k$ exist and are continuous within and on the boundary of the region D considered,* $(i, k = 1, 2, \ldots, m)$.

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* Goursat, *Cours d'Analyse*, vol. 3, 2d edition, p. 13.

APPLICABILITY WITH PRESERVATION OF BOTH CURVATURES*

BY W. C. GRAUSTEIN

1. *Introduction.* In his celebrated memoir of 1867, Bonnet† proved that there is in general no surface applicable to a given surface with preservation of both the total and mean curvatures, but that in certain exceptional cases there exists a unique surface or a one-parameter family. The primary purpose of this note is to exhibit conditions of simple form characterizing the second of these exceptional cases. The method of treatment applies equally well to the general case.

2. *Necessary and Sufficient Condition.* It is well known that a necessary condition that a surface S admit ∞^1 surfaces applicable to it with preservation of both curvatures is that the surface S be isometric.‡ Accordingly, we can assume that S is an isometric surface, referred to its lines of curvature, and that the parameters are isometric. The linear element of S is then of the form

$$ds^2 = \lambda(du^2 + dv^2), \tag{1}$$

* Presented to the American Mathematical Society, September 7, 1923.

† *Mémoire sur la théorie des surfaces applicables sur une surface donnée,* JOURNAL DE L'ÉCOLE POLYTECHNIQUE, vol. 42 (1867), pp. 72 et seq.

‡ This fact appears to have been discovered first by Raffy, *Sur une classe nouvelle de surfaces isothermiques et sur les surfaces déformables sans alteration des courbures principales,* BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 21 (1893), pp. 70–72. His proof, however, is faulty, in that he bases part of it on the incorrect statement, made by Caronnet, that every W-surface applicable to a surface of revolution is isometric. If that statement were correct, every helecoidal surface would be isometric, which is not the case. A valid proof is given by Hazzidakis, *Biegung mit Erhaltung der Hauptkrümmungsradien,* JOURNAL FÜR MATHEMATIK vol. 117 (1897), p. 46.

and the Codazzi equations become

$$(2) \qquad \frac{\partial e}{\partial v} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial v}, \quad \frac{\partial g}{\partial u} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial u},$$

where $e, f\,(=0), g$ are the differential coefficients of the second order. Recalling that $1/r_1 = e/\lambda$, $1/r_2 = g/\lambda$, and setting

$$\frac{1}{r_1}+\frac{1}{r_2} = 2M, \quad \frac{1}{r_1}-\frac{1}{r_2} = 2N,$$

we find that equations (2) can be rewritten in the forms*

$$(3) \qquad \frac{\partial M}{\partial u} = N\frac{\partial \log \lambda N}{\partial u}, \quad \frac{\partial M}{\partial v} = -N\frac{\partial \log \lambda N}{\partial v}.$$

If there exists a surface S' applicable to S with preservation of both curvatures, then

$$e'g'-f'^2 = eg, \quad e'+g' = e+g.$$

Hence it can be shown that

$$(4) \qquad e' = e-\frac{2\lambda N}{1+z^2}, \quad f' = \frac{2\lambda N z}{1+z^2}, \quad g' = g+\frac{2\lambda N}{1+z^2},$$

where

$$z = \operatorname{ctn}\alpha,$$

and where α is the angle under which the curves corresponding to the lines of curvature on S' cut the lines of curvature on S.

The Codazzi equations for S', by virtue of the corresponding equations (2) for S, can be written in the forms

$$\frac{\partial(e'-e)}{\partial v} = \frac{\partial f'}{\partial u}, \qquad \frac{\partial(g'-g)}{\partial u} = \frac{\partial f'}{\partial v}.$$

Substituting for $e'-e$, f', and $g'-g$ their values as given by (4), and solving for $\partial z/\partial u$, $\partial z/\partial v$, we obtain two equations which reduce to

$$(5) \qquad \frac{\partial}{\partial u}\frac{z}{\lambda N} = \frac{\partial}{\partial v}\frac{1}{\lambda N}, \qquad \frac{\partial}{\partial v}\frac{z}{\lambda N} = -\frac{\partial}{\partial u}\frac{1}{\lambda N}.$$

* We have hereby assumed that $N \neq 0$. But a sphere obviously admits no surface applicable to it in the manner prescribed.

But these equations are compatible if and only if the reciprocal of λN is a harmonic function.

THEOREM 1. *There exists a one-parameter family of surfaces applicable to a given surface S with preservation of both curvatures if and only if the lines of curvature on S form an isometric system, and if, when isometric parameters for this system are introduced, the reciprocal of* $\lambda(1/r_1 - 1/r_2)$ *is a harmonic function.*

If one function, ψ, to which the reciprocal of λN is conjugate, is obtained by a quadrature, the general solution of equations (5) is

$$z = \lambda N(\psi + k),$$

where k is an arbitrary constant. Substituting this value of z in (4), we obtain e', f', and g' for the surfaces S'. Thus these surfaces are completely determined.

That a surface S of constant mean curvature admits ∞^1 surfaces S' applicable to it with preservation of both curvatures is well known.*

It is in this case only that the angle α is constant and that the lines of curvature on S correspond to the lines of curvature on one of the surfaces S'. For, by (3), the mean curvature is constant only if λN is constant, and, by (5), λN is constant only when z is constant.

3. *Applicability in an Infinity of Ways.* Consider now an arbitrary one-parameter family of surfaces, every pair of which are applicable with preservation of both curvatures. An obvious necessary condition that this applicability be possible in a continuous infinity of ways is that the surfaces be W-surfaces, i. e., that the curvatures be functionally related. It is surprising to find that, if we exclude the surfaces of constant mean curvature, this condition is also sufficient.

THEOREM 2. *Every pair of a one-parameter family of (isometric) W-surfaces satisfying the conditions of Theorem* 1, *but of variable mean curvature, are applicable in a continuous infinity of ways with preservation of both curvatures.*

* Cf. Bonnet, loc. cit.

For, if we except surfaces of constant mean curvature, an isometric W-surface is either a surface of revolution or a surface of one of several special types. The theorem is obvious in the first case. In the second case, it follows directly from certain investigations by the author, which are to be published elsewhere.

A surface of constant mean curvature admits ∞^1 surfaces each applicable to it in a continuous infinity of ways with preservation of both curvatures if and only if it is applicable to a surface of revolution. The surfaces of constant mean curvature with this property have been determined by Raffy.*

4. *Conclusion.* In conclusion, we sketch the application of the method of § 2 to the general problem. In this case we assume merely that the lines of curvature on S are parametric. If S' is applicable to S with preservation of both curvatures,

$$e'g' - f'^2 = eg, \qquad Eg' + Ge' = Eg + Ge.$$

Then

$$e' = e - \frac{2EN}{1+z^2}, \quad f' = \frac{2\sqrt{EG}\,Nz}{1+z^2}, \quad g' = g + \frac{2GN}{1+z^2},$$

where $z = \operatorname{ctn}\alpha$, and α has the same significance as before.

The Codazzi equations for S', after application of those for S, become

$$\frac{\partial(e'-e)}{\partial v} = f'\frac{\partial}{\partial u}\log\left(f'\sqrt{\frac{G}{E}}\right),$$

$$\frac{\partial(g'-g)}{\partial u} = f'\frac{\partial}{\partial v}\log\left(f'\sqrt{\frac{E}{G}}\right).$$

Proceeding as in § 2, we obtain finally the equations

$$(6) \qquad \frac{\partial z}{\partial u} = Az - B\sqrt{\frac{E}{G}}, \qquad \frac{\partial z}{\partial v} = Bz + A\sqrt{\frac{G}{E}},$$

* *Sur certaines surfaces dont les rayons de courbure sont liés par une relation,* BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 89 (1890-91), pp. 158-169.

where

$$A = \frac{1}{N}\frac{\partial M}{\partial u} = \frac{\partial \log GN}{\partial u},$$
$$B = -\frac{1}{N}\frac{\partial M}{\partial v} = \frac{\partial \log EN}{\partial v}. \tag{7}$$

But equations (6) are compatible if and only if

$$Pz = Q,$$

where

$$\begin{cases} P = \dfrac{\partial A}{\partial v} - \dfrac{\partial B}{\partial u} = \dfrac{\partial^2}{\partial u \partial v}\log\dfrac{G}{E}, \\ Q = \dfrac{\partial}{\partial u}\left(A\sqrt{\dfrac{G}{E}}\right) + \dfrac{\partial}{\partial v}\left(B\sqrt{\dfrac{E}{G}}\right) - A^2\sqrt{\dfrac{G}{E}} - B^2\sqrt{\dfrac{E}{G}}. \end{cases} \tag{8}$$

THEOREM 3. *There are ∞^1 surfaces S' applicable to a given surface S with preservation of both curvatures if $P = 0$, $Q = 0$; there is a unique surface S' if $P \neq 0$ and $z = Q/P$ satisfies equations* (6). *Otherwise no surface S' exists.*

According to (8), the vanishing of P is the condition that S be an isometric surface. If, then, isometric parameters are introduced, so that $E = G = \lambda$, $Q = 0$ becomes the condition that $1/\lambda N$ be a harmonic function, provided that for A and B the second values given in (7) are used. If the first values are substituted instead, $Q = 0$ reduces to a condition more complicated in form, but involving merely the principal radii of curvature.

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COMPLETE SETS OF REPRESENTATIONS OF TWO-ELEMENT ALGEBRAS*

BY B. A. BERNSTEIN

1. *Introduction.* Any algebra consists of one or more classes of elements and one or more operations or relations among the elements. An operation $\oplus$ of a class K of two elements, a, b is given by a $\oplus$-table of the form

$\oplus$	a	b
a	c_1	c_2
b	c_3	c_4

where the c's all belong to K, or do not all belong to K, according as the operation is K-closing or not. A relation R in K is given by an R-table of the form

R	a	b
a	$\pm$	$\pm$
b	$\pm$	$\pm$

where the sign $+$ indicates that $a\,R\,b$ holds, the sign $-$ that $a\,R\,b$ does not hold. The object of this paper is to give convenient representations of these $\oplus$-tables and R-tables, hence of all two-element algebras, and to point out the usefulness of these representations in connection with fundamental questions relating to postulate-sets.

2. *Representation of Class-closing Operations.* There are 2^4 operations $\oplus$ possible in a class of two elements when the condition of closure is satisfied. The following theorem gives us two sets of representations of these operations, one arithmetic and one boolean. In the boolean representation the symbols 0, 1, a', $a+b$, ab denote respectively

* Read before the San Francisco Section of the Society April 7, 1923. The paper includes the substance of the following papers, presented before the San Francisco Section October 21, 1922:

(1) *An arithmetic representation of boolean logic,* (2) *Arithmetic independence systems for the Whitehead-Huntington postulates for boolean algebras,* (3) *A boolean representation of a number field.*

the logical "zero," the "whole," the negative of a, the logical sum of a and b, the logical product of a and b.

THEOREM A. *The* 2^4 *systems* $(K, \oplus)$ $(K: 0, 1;\ a \oplus b = 0, 1)$ *are equivalent to the arithmetic functions*

$$\text{(I)} \qquad A_1ab + A_2a + A_3b + A_4 \text{ modulo } 2, \quad (A_i = 0, 1)$$

and also to the boolean functions

$$\text{(II)} \qquad Aab + Bab' + Ca'b + Da'b', \quad (A, B, C, D = 0, 1).$$

The theorem is true for the following reasons:

(1) Every system $(K, \oplus)$ of the theorem is equivalent to a $\oplus$-table of the form

(III)

$\oplus$	0	1
0	e_1	e_2
1	e_3	e_4

$(e_i = 0, 1).$

(2) Every function of form (I) or of form (II) determines a $\oplus$-table (III).

(3) Every $\oplus$-table (III) determines a function $f(a, b)$ of form (I) from the equations (modulo 2)

$$f(0, 0) = A_4, \qquad f(0, 1) = A_3 + A_4,$$
$$f(1, 0) = A_2 + A_4, \qquad f(1, 1) = A_1 + A_2 + A_3 + A_4.$$

(4) Every $\oplus$-table (III) determines a function $F(a, b)$ of form (II) from the fact that

$$F(0, 0) = D, \quad F(0, 1) = C, \quad F(1, 0) = B, \quad F(1, 1) = A.$$

Table A gives the representations determined by Theorem A.

3. *Representation of Relations.* Since functions (I), as well as functions (II), of Theorem A are all distinct, we have the following theorem.

THEOREM B. *The* 2^4 *systems* (K, R) $(K: 0, 1;\ R$ *a dyadic relation) are equivalent to the arithmetic equations* (*modulo* 2)

$$\text{(I')} \qquad A_1ab + A_2a + A_3b + A_4 = 0, \quad (A_i = 0, 1),$$

and also to the boolean equations

$$\text{(II')} \quad Aab + Bab' + Ca'b + Da'b' = 0, \quad (A, B, C, D = 0, 1).$$

Table B gives the representations determined by Theorem B.

TABLE *A*

REPRESENTATIONS OF SYSTEMS $(K, \oplus)$
$(K: 0, 1;\ a \oplus b = 0, 1)$

No.	$a \oplus b$	Arithmetic Representation	Boolean Representation
1	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$	0	0
2	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$	$ab \pmod 2$	ab
3	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}$	$ab+b \pmod 2$	$a'b$
4	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}$	b	$ab+a'b=b$
5	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}$	$ab+a \pmod 2$	ab'
6	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}$	a	$ab+ab'=a$
7	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$	$a+b \pmod 2$	$ab'+a'b$
8	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$	$ab+a+b \pmod 2$	$ab+ab'+a'b=a+b$
9	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}$	$ab+a+b+1 \pmod 2$	$a'b'$
10	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$	$a+b+1 \pmod 2$	$ab+a'b'$
11	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}$	$a+1 \pmod 2$	$a'b+a'b'=a'$
12	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$	$ab+a+1 \pmod 2$	$ab+a'b+a'b'=(ab')'$
13	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}$	$b+1 \pmod 2$	$ab'+a'b'=b'$
14	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 1 \end{array}$	$ab+b+1 \pmod 2$	$ab+ab'+a'b'=(a'b)'$
15	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}$	$ab+1 \pmod 2$	$ab'+a'b+a'b'=(ab)'$ $=a'+b'$
16	$\begin{array}{c\|cc} _ & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$	1	1

TABLE *B*

REPRESENTATIONS OF SYSTEMS (K, R)
(K: 0, 1; R A DYADIC RELATION)

No.	$a\ R\ b$	Arithmetic Representation	Boolean Representation
1	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & - \end{array}$	$1=0$	$1=0$
2	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & + \end{array}$	$ab+1=0 \pmod 2$	$ab'+a'b+a'b'=(ab)'$ $=a'+b'=0$
3	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}$	$ab+b+1=0 \pmod 2$	$ab+ab'+a'b'=(a'b)'=0$
4	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & + \end{array}$	$b+1=0 \pmod 2$	$ab'+a'b'=b'=0$
5	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & - \end{array}$	$ab+a+1=0 \pmod 2$	$ab+a'b+a'b'=(ab')'=0$
6	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & + \end{array}$	$a+1=0 \pmod 2$	$a'b+a'b'=a'=0$
7	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & - \end{array}$	$a+b+1=0 \pmod 2$	$ab+a'b'=0$
8	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & + \end{array}$	$ab+a+b+1=0 \pmod 2$	$a'b'=0$
9	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & - \end{array}$	$ab+a+b=0 \pmod 2$	$ab+ab'+a'b=a+b=0$
10	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & + \end{array}$	$a+b=0 \pmod 2$	$ab'+a'b=0$
11	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & - \end{array}$	$a=0$	$ab+ab'=a=0$
12	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & + \end{array}$	$ab+a=0 \pmod 2$	$ab'=0$
13	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & - \end{array}$	$b=0$	$ab+a'b=b=0$
14	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & + & + \end{array}$	$ab+b=0 \pmod 2$	$a'b=0$
15	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & - \end{array}$	$ab=0 \pmod 2$	$ab=0$
16	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & + \end{array}$	$0=0$	$0=0$

4. *Representation of Operations that are not Class-closing.* Representations of operations $\oplus$ that do not satisfy the condition of closure are given by the following theorem.

THEOREM *C*. *Let* $\sum \equiv (K, \oplus)$ *be a system of two elements* 0, 1, *with* $\oplus$ *such that* $a \oplus b = x^*$ *not in* K *for some values of* a, b; *let* $f(a, b)$ *be an arithmetic function equivalent* (*Theorem A*) *to a system obtained from* $\sum$ *by replacing the* x*'s by* 0 *or* 1; *and let* $\Phi(a, b)$ *be an arithmetic function equivalent to the system obtained from* $\sum$ *by replacing* (1) *the* x*'s by* 0 *and* (2) *the non-*x*'s by* 1; *then the function*

$$F(a, b) \equiv f(a, b) + \frac{0}{\Phi(a, b)}$$

is an arithmetic equivalent of $\sum$.

For $F(a, b) = f(a, b) + (0/1)$ when a, b are such that $a \oplus b = 0$ or 1 in $\sum$; and $F(a, b) = f(a, b) + (0/0)$ when a, b are such that $a \oplus b = x$ in $\sum$.

By defining a/b as the (unique) element y which satisfies the boolean equation $by = a$, we can obtain boolean representations of systems $\sum$ of Theorem *C*. Since 0/0 is not an element of a boolean algebra, we have

THEOREM *C'*. (The same as Theorem *C* with the word *arithmetic* everywhere replaced by the word *boolean*.)

Thus, to find a representation of the system $\sum \equiv (K, \oplus)$ determined by the table

$\oplus$	0	1
0	0	1
1	1	x

(x not in K),

take from Table A system 7† for $f(a, b)$ and system 15 for $\Phi(a, b)$. An arithmetic representation of $\sum$ is then

$$a + b \pmod 2 + \frac{0}{ab + 1 \pmod 2},$$

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* I am using the expression "$a \oplus b = x$ not in K" for some values a, b, to mean that for the values a, b in question $a \oplus b$ is meaningless.

† Or also system 8.

and a boolean representation is

$$ab' + a'b + \frac{0}{a' + b'}.$$

5. *Applications.* The representations of tables A, B obviously have a bearing on postulational questions. I point out here a few facts obtained with the help of the tables.

(1) A two-element boolean algebra $(K, \oplus, \odot)$ may be represented by an arithmetic system; namely, by

$$K{:}0, 1;\ a \oplus b = ab + a + b \pmod 2;\ a \odot b = ab \pmod 2,$$

or, dually, by

$$K{:}0, 1;\ a \oplus b = ab \pmod 2;\ a \odot b = ab + a + b \pmod 2.$$

(2) The two-element boolean algebra $(K, <)$, expressed in terms of the relation $<$ of inclusion, may be represented by the arithmetic system

$$K{:}0, 1;\quad a < b = ab + a = 0 \pmod 2.$$

(3) Each of the $\oplus$-tables 7 and 10 of Table A satisfies the conditions for an abelian group.* Hence, the logical elements 0, 1 form an abelian group with respect to each of the operations

$$ab' + a'b, \qquad ab + a'b'.\dagger$$

(4) In Table A, $\oplus$-tables 7 and 2 together define a field $(K, \oplus, \odot)$.‡ Hence the logical elements 0, 1 form a field with respect to the pair of operations $ab' + a'b$, ab, and also with respect to the dual pair $ab + a'b'$, $a + b$.

(5) All boolean operations can be defined in terms of each of the operations $a'b'$, $a' + b'$.§ Hence all the arith-

* For postulates for an abelian group see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 27.

† It can be shown that the totality of the elements of *any* boolean algebra, finite or infinite, form an abelian group with respect to each of these operations.

‡ For postulates for fields see TRANSACTIONS OF THIS SOCIETY, vol. 4 (1903), p. 31.

§ See Sheffer's postulates for boolean algebras, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 481.

metic operations of Table A can be defined in terms of each of the operations $ab + a + b + 1 \pmod 2$, $ab + 1 \pmod 2$. And any two-element algebra can be defined by means of an appropriate postulate-set involving the single operation 9 or the single operation 15.

(6) Huntington's abstract systems $(K, \oplus, \odot)$ proving the independence of his first set of postulates for boolean algebras* may be replaced by the arithmetic systems† of the following table (Systems I'_a, I'_b, ..., VI' are independence systems respectively for postulates I_a, I_b, ..., VI).

TABLE C

	K	$a \oplus b$	$a \odot b$
I'_a	0,1	$a+b \pmod 2 + \dfrac{0}{ab+1 \pmod 2}$	ab
I'_b	0,1	ab	$a+b \pmod 2 + \dfrac{0}{ab+1 \pmod 2}$
II'_a	0,1	0	ab
II'_b	0,1	ab	0
III'_a	0,1	a	ab
III'_b	0,1	ab	a
IV'_a	0,1	$a + b \pmod 2$	ab
IV'_b	0,1	ab	$a + b \pmod 2$
V'	0,1	$ab + a + b \pmod 2$	$ab + a + b \pmod 2$
VI'	0	$0 + 0$	$0 \cdot 0$

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* See TRANSACTIONS OF THIS SOCIETY, vol. 5 (1904), p. 288.

† With the help of Table A we can, of course, write down boolean systems for Huntington's abstract systems.

BROUWER'S CONTRIBUTIONS TO THE FOUNDATIONS OF MATHEMATICS*

BY ARNOLD DRESDEN

1. *Introduction.* In a number of papers, published from 1907 on, Professor L. E. J. Brouwer, of the University of Amsterdam, has developed ideas which affect the foundations of mathematics in a fundamental way. While some of his papers are readily available to American mathematicians,† there are several others which are less accessible. On account of its critique of some of our most fundamental concepts and methods, the position of Brouwer may have a far reaching effect upon the future development of mathematics. In his *Begründung der Mengenlehre*, he has made a beginning with a revision of a basic field of modern mathematics in accordance with his point of view. But, whatever their ultimate significance may be, the conclusions which Brouwer reaches are certainly interesting. Moreover, they are indispensable as a background for an appreciation of his *Begründung der Mengenlehre*, as well as for understanding the controversial discussions on the foundations of mathematics of Weyl and Hilbert.‡

For this reason, it has seemed worth while to present Brouwer's most important ideas concerning the foundations of mathematics to American readers. For this purpose, we

* Presented to the Society, December 29, 1923.

† See *Intuitionism and formalism*, this BULLETIN, vol. 20 (1913), p. 81; Review of Schoenflies-Hahn, *Die Entwickelung der Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), p. 78; *Intuitionistische Mengenlehre*, JAHRESBERICHT DER VEREINIGUNG, vol. 28 (1920), p. 203; *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*, Amsterdam, 1918–19.

‡ See Weyl, *Über die neue Grundlagenkrise der Mathematik*, MATHEMATISCHE ZEITSCHRIFT, vol. 10 (1921), p. 39; Hilbert, *Neubegründung der Mathematik*, ABHANDLUNGEN DER HAMBURGISCHEN UNIVERSITÄT, vol. 1 (1922), p. 157, and *Die logischen Grundlagen der Mathematik*, MATHEMATISCHE ANNALEN, vol. 88 (1922), p. 151.

have used, besides the material referred to above, his book on the foundations of mathematics, (*Over de Grondslagen der Wiskunde,* Amsterdam, Maas & van Suchtelen, 1907) and his article on the unreliability of logical principles (*De onbetrouwbaarheid der logische principes,* TIJDSCHRIFT VOOR WIJSBEGEERTE, vol. 1 (1908)). Our discussion falls into three parts, viz., mathematics and experience, mathematics and mathematical language, mathematics and logic.

2. *Mathematics and Experience.* Brouwer conceives of mathematical thinking as a process of construction, which builds its own universe, independent of the universe of our experience, somewhat as a free design, under the control of nothing but arbitrary choice, restricted only in so far as it is based upon the fundamental mathematical intuition. This intuition, upon which not only mathematical thinking, but all intellectual activity is held to be based, is found in the abstract substratum of all observation of change, "a fusion of continuous and discrete, a possibility of conceiving simultaneously several units, connected by a '*between*' that cannot be exhausted by the interpolation of new units."

It is not to be expected that all mathematicians will agree with this point of view. It is in this conception of the source and of the character of mathematical thinking that the ideas of Brouwer have their root. Its relation to the thought of Plato* and of other Greek philosophers, interesting as it is, must be left untouched here except for the observation that the acceptance of this union of discrete and continuous as the rock bottom of mathematical thinking disposes of the paradoxes of Zeno and of the conflicts of Parmenides somewhat as the theory of relativity disposed of the drag of the ether.†

In a similar way it disposes of many questions in point set theory, which have occupied the attention of mathematicians. For, by combining continuous and discrete in one fundamental concept, it renders futile all attempts at building up one of

* See, e. g., Brunschvieg, *Les Étapes de la Philosophie Mathématique,* p. 49 et seq.

† Compare Brunschvieg, loc. cit., p. 155.

these by means of the other, conceived as independent of the first. But these matters have been dealt with extensively in *Intuitionism and Formalism.*

The fundamental intuitive concept of mathematics must not be thought of as in the nature of an undefined idea, such as occur in postulational theories, but rather as something in terms of which all undefined ideas which occur in the various mathematical systems are to be intuitively conceived, if they are indeed to serve in mathematical thinking. It manifests itself in the intuition of time, which makes possible "repetition, as being object in time and again object."

The position of Brouwer on this point is directly opposed to those of Kant and of Russell, who hold that mathematical thinking cannot be based on the one-dimensional time continuum alone, but that it requires also three-dimensional euclidean space (Kant, *Transcendental Ethics*), or projective space (Russell, *Foundations of Geometry*).

In the first chapter of *Over de Grondslagen der Wiskunde,* Brouwer constructs on the basis of this fundamental intuition the order types ω and η, and the elementary propositions of algebra and geometry. In these building processes, experience plays no part, and Brouwer holds that "in this constructive process, bound by the obligation to notice with care which theses are acceptable to the intuition and which are not, the only possible foundation for mathematics is to be looked for."*

It must of course be remembered that these statements concerning the role of experience are to be taken in the philosophical sense, not in the historical sense. For no one could deny that in the historical development of mathematics, experience played a permanent part. On the other hand, one will have to admit that, while "experimental *science* is linked up with mathematics, experience can never force the

* It may be of interest to compare with this statement, the one found on p. 50 of Boutroux, *L'Idéal Scientifique des Mathématiciens,* in a discussion of the Hellenic conception: "And if it frequently happens that we make mistakes, it is because we have obscured our vision by insufficient exercise of our intuitive faculty."

choice of a *particular* mathematical system." The question as to the role of experience in the development of mathematics seems to the present author still to have a good many aspects that call for further study. Enough has been said however to indicate Brouwer's fundamental thesis and to discuss some of its important consequences.

Two other points need however still to be made clear. It must already have become evident to the reader that Brouwer is not seeking to build up a system of postulates for mathematics, either in whole or in part. The freedom of choice in the construction of mathematics, once the fundamental intuition is recognized, leaves the way open for setting up various postulate systems for any part of mathematics or for the whole science. This seems indeed to be the most reasonable attitude towards postulational theory. Each system of postulates for a particular field of knowledge is to be looked upon as a set of pronouncements in terms of undefined ideas, which are verifiable in that particular field, if the undefined ideas are suitably particularized. In the measure in which these postulates are independent they enlarge our knowledge of the structure of the field; in the measure in which they are non-categorical, they establish relations with other fields.

In the second place, even though constructed without any direct interplay of experienced reality, mathematics is not without value for practical life. Because through the agency of mathematical building, phenomena are linked together in causal sequences, which enable man to control the external world. "The conduct of man aims to observe as many as possible of these mathematical sequences, in order that, whenever an earlier element in such a sequence offers in actuality a better opportunity for taking hold of the situation than a later element in the same sequence, even though only the later one appeals to his instincts, he may choose the earlier one as the object of his acts." The mathematical universe thus becomes an accompaniment of the phenomenal universe, which assists man in his control of the latter.

3. *Mathematics and Mathematical Language.* The relation of Brouwer's thought to the Platonic point of view, hinted at above, is brought out once more in his insistent separation of mathematics from the language of mathematics. In Book VII of *The Republic*, Socrates is made to say, that "on one point at any rate we shall encounter no opposition from those who are even slightly acquainted with geometry, when we assert that this science holds a position which flatly contradicts the language employed by those who handle it." In quoting this passage and in commenting upon it, Boutroux gives more emphatic utterance to its thought: "It is known indeed, that the Platonists established a profound distinction between 'discourse' and 'intelligence', between written science, which is a didactic exposition of truths already known, and the *conception* of scientific truths, which is the direct product of our faculty of intuition in its dealing with the world of ideas."* For Brouwer, mathematics is a process of construction, and "of the mathematical building and reasoning, and in particular of the logical reasoning which men do within themselves, they try to evoke copies in other men by means of sounds and symbols, which also serve to aid their own memory." It seems that creative mathematicians cannot but receive with approval Brouwer's remark, that "in arguments concerning experiential realities, fitted into mathematical systems, logical principles are not the guide, but rather a regularity observed a posteriori in the accompanying language; and if one speaks in accordance with this regularity, but detached from mathematical systems, there is always a danger of paradoxes, like that of Epimenides." Mathematical proof without the use of words consists in establishing relations between different parts of the mathematical edifice, i. e. "when mathematical objects are given by means of their relations to elements or fragments of a mathematical edifice, one transforms these relations by a series of tautologies and thus one progresses step by step to the relations of the objects with other parts of the edifice."

* P. Boutroux, loc. cit., p. 35.

It is only in the language which accompanies this process for purposes of communication and memory, that logical forms arise. "The words of your mathematical demonstration are but the accompaniment of wordless mathematical building", Brouwer says to the logician, "and when you establish a contradiction, I simply observe that the construction cannot go on, that in the given edifice no room can be found for the posited structure. And when I make this observation I do not think of the Law of Contradiction." It is clear that the role of logic, as here conceived, is very different from the one usually attributed to it.

Before taking up more fully Brouwer's views of the relation of mathematics to logic, it will be of interest to insert the following passage: "The mathematical fact is independent of the logical or algebraic dress in which we seek to represent it; indeed, the idea which we have of it is richer and fuller than all the definitions which we can give of it, than all the forms or combinations of signs or of propositions by means of which we can express it. The expression of a mathematical fact is arbitrary, conventional. But the fact itself, that is to say, the truth which it contains, forces itself upon our mind apart from all conventions. Thus, one could not account for the development of mathematical theories, if one tried to consider the algebraic formulas and the logical combinations as the objects whose study the mathematician pursues. However, all the characteristics of these theories are easily explained, once one admits that the algebra and the logical propositions are but the language into which one *translates* a set of ideas and of objective facts."*

4. *Mathematics and Logic.* Indeed, Aristotelian logical reasoning is but a special kind of mathematical reasoning, namely that kind which is "concerned exclusively with relations of 'whole and part'." And the language which accompanies such logical reasoning is the *language of logical reasoning*, just as *mathematical language* is that which

* P. Boutroux, loc. cit., p. 203.

accompanies mathematical reasoning. Furthermore, these languages, themselves, like other parts of the phenomenal world can become the object of mathematical observation and study; thus arise *theoretical logic* as the mathematics of the language of logical reasoning, and *logistics* as the mathematics of the language of mathematical reasoning.

In view of these characterizations, it is not surprising to find little sympathy with the attempts to lay down logical foundations for mathematics. "A logical building up of mathematics, independent of the mathematical intuition is impossible — because in this way we obtain but a verbal edifice irrevocably apart from mathematics proper — and moreover a contradiction in terms, because a logical system, as well as mathematics itself, requires the fundamental intuition of mathematics."

The fundamental difference between the point of view of Brouwer concerning the nature of mathematics and that of Hilbert, as expressed in the latter's *Neubegründung*, referred to above, comes out clearly in the following sentences: "Suppose we have proved by some method or other, without having a mathematical interpretation in mind, that a logical system built up on the basis of some verbal axioms, is non-contradictory, i. e., that at no point of the development of the system two contradictory propositions will arise; and suppose that we then find a mathematical interpretation of the axioms, (which consists of requiring a construction of a mathematical edifice from elements which satisfy given mathematical relations). Does it then follow from the non-contradictoriness of the logical system that such a mathematical structure exists? No such thing has ever been proved by the postulationists" . . . "so, e. g., it has nowhere been proved, that if a finite number must satisfy a set of conditions which can be shown to be non-contradictory, that then this number actually exists."* If we compare this paragraph with Hilbert's system of undefined ideas and of postulates for mathematics, one is reminded of the phrase

* *Grondslagen*, p. 141.

of Poincaré, quoted elsewhere by Brouwer,* "Les hommes ne s'entendent pas, parce qu'ils ne parlent pas la même langue et qu'il y a des langues qui ne s'apprennent pas."

Concerning the logical paradoxes which have disturbed some mathematicians during the last twenty-five years and the attempts to resolve them by means of more refined logical methods, Brouwer holds that "they arise whenever regularity in the language which accompanies mathematics is extended so as to apply also in a language of mathematical words, which do not accompany mathematics." Moreover, "logistics concerns itself with mathematical language, instead of with mathematics, and consequently does not clarify mathematics; finally all paradoxes disappear if one restricts oneself to dealing with systems that are explicitly constructible on the basis of the fundamental intuition," i. e., if one gives priority to mathematics instead of to logic. This aspect of Brouwer's position again finds support elsewhere: "In other words, the most important advances which mathematicians make, are obtained not in perfecting the form, but in modifying the basis of the theory. These advances cannot be regarded as being of logical character."† . . . "In order to give mathematical theories a firm structure, we have decided to give them the form of logical systems; but, observing that these systems are artificial and can moreover be infinitely diversified, we realize that they neither constitute the whole of mathematics, nor its principal part. Behind the logical form there is something else."‡

But Brouwer does not merely indulge in a general criticism of the role of logic in mathematics; he proceeds to a discussion of the profoundly important question: "Is it allowable, in dealing with purely mathematical constructions and transformations temporarily to neglect the idea of the constructed mathematical system and to work with the accompanying verbal structure, guided by the principles of the *syllogism*,

* See this BULLETIN, vol. 20 (1913), p. 96.

† P. Boutroux, loc. cit., p. 168.

‡ Ibid. p. 170.

of *contradiction*, and of the *excluded middle*, confident that by evoking temporarily the idea of the reasoned mathematical constructions, every part of the argument could be validated in turn?"*

It should be clear that for the actual work of mathematical research this question, once one adopts Brouwer's conception of the character of mathematical thinking, is of primary importance. And his answer is *Yes*, as concerns the principles of the syllogism and of contradiction, but *No* for the law of the excluded middle. While the law of contradiction asserts that it is impossible for a proposition to be both true and false, the law of the excluded middle (L. E. M.) says that every proposition is either true or false. Its acceptance leads therefore to a belief in the solvability of every mathematical problem.† For, from Brouwer's point of view, this principle asserts that "for every hypostatized fitting into each other of systems in a definite way, either the actual construction can be made, or an insurmountable obstruction can be erected." If the proposition deals with fragments of a finite, definite, discrete system, this possibility will readily be granted, so that the L. E. M. may be considered as valid in dealing with such cases. For instance, of two positive integers, it can be affirmed that either they are relatively prime, or they possess a common divisor different from unity.

But the situation becomes different when we are dealing with infinite systems. Propositions concerning infinite systems can be dealt with systematically only when the use of complete induction is possible; in such a case the infinite system can be fitted in by the use of properties of an *arbitrary element*. On the other hand, the totality of the mathematical properties and contradictions derivable by means of complete induction forms what Brouwer calls a "denumerably un-

* A statement of the laws of thought will be found in any text on formal logic; see, e. g. Jevons, *Elementary Lessons in Logic*, p. 117.

† Compare Hilbert, *Mathematische Probleme*, GÖTTINGER NACHRICHTEN, 1900.

finished set," i. e., a set of which nothing but a denumerable subset can ever be explicitly exhibited, and such that whenever a denumerable subset is given, a new element of the set can always be derived from it by means of a previously defined process.* The possibility of systematically establishing the truth or falsity of a proposition concerning an arbitrarily proposed infinite system depends therefore upon finding among the denumerably unfinished set of mathematical properties and contradictions one which (eventually by means of complete induction, i. e. "by means of an element invariant over a denumerably infinite sequence") enables us to put the proposition as one which can be dealt with and decided, one way or the other, by the use of complete induction.

But the search for such a structure of property or contradiction cannot be carried out systematically; hence its success depends more or less upon good luck and cannot be assured a priori. Hence it is uncertain whether for an arbitrary proposition concerning a given infinite system either the construction or the obstruction can be established, and hence it is equally uncertain whether the L. E. M. is valid in such a case. But, still further, unjustified assumption that one or the other must be possible can never be detected; for that would mean that both the hypothesis of construction and that of obstruction would lead to an obstruction in the further process of construction, which conflicts with the law of contradiction.

It is on the basis of these considerations that Brouwer denies unlimited validity to the L. E. M., and that he reaches the following conclusion: "In mathematics, it is not certain whether or not all logic is permissible, and it is not certain whether it can be decided, whether or not all logic is permissible."

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* This notion, of which the set of well-ordered ordinals is an example, plays an important part in much of Brouwer's work.

A QUALITATIVE DEFINITION OF THE POTENTIAL FUNCTIONS*

BY PHILIP FRANKLIN

1. *Introduction.* In this paper we aim to set up postulates completely characterizing the potential functions, which do not involve derivatives or integrals, and are thus of a more qualitative nature than the definitions previously given. Incidentally, we may interpret all of our postulates as statements of properties of such physical quantities as give rise to potential functions, and when so interpreted, we see from physical grounds that they hold for the quantities in question. They thus furnish a means of going directly from certain physical problems to the potential functions, without the use of Laplace's equation. Our results will be stated in full only for potential functions of two and three variables, although they may evidently be extended to the n-dimensional case.

2. *Postulates for Two Dimensions.* Consider a class of functions of two variables, x and y, thought of for convenience as Cartesian coordinates, and let each function have associated with it a region R of the plane. Our assumptions are:

(1) Each function is continuous in both variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y)$ and $f_2(x, y)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y) + Bf_2(x, y)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y)$ with region R into $F(x', y')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

* Presented to the Society, April 28, 1923.

(4) The function $f(x, y) = 1$ is a function of the class, its region R being the entire plane.

(5) If a given circle lies entirely inside each region R_i, belonging to $f_i(x, y)$, these $f_i(x, y)$ constituting an infinite sequence of functions of the class, and if, for points x_c, y_c on the circumference of the circle $\lim_{i \to \infty} f_i(x_c, y_c) = 0$, the limit being approached uniformly over the circumference, then the sequence of values of the functions $f_i(x, y)$ at the center of the circle cannot approach a limit different from zero.

We shall show that any family of functions satisfying these five postulates is necessarily a class of potential functions, each function being harmonic at any interior point of its region R. Since every linear class of harmonic functions satisfies the above postulates, the widest class of functions satisfying them is the totality of harmonic functions.

By way of motivating our choice of postulates, we note that postulates 1, 2, and 4 are natural requirements since the functions we are trying to define satisfy a linear homogeneous partial differential equation. Postulate 3 is allied to the fact that the Laplacian operator is the only linear differential operator of the second order invariant under orthogonal transformations of coordinates. The last postulate is added as a fairly weak condition which completes the characterization, since the preceding ones are not by themselves sufficient for this.

It is illuminating to consider the significance of the postulates for a physical example, say the functions giving the temperature of points of a set of thin plates, when various boundary temperatures are assigned. The postulates state, essentially, that these functions are continuous point functions, as a class independent of the choice of coordinates, which possess the property of combination by superposition; that if the boundary temperatures are constant, the temperature inside will be this same constant, and finally that if the temperatures at the circumference of a circular plate are altered so that they approach zero uniformly, the temperature at the center of the circular plate will approach zero.

3. *Deduction of the Mean-Value Property.* We shall now prove that the class of functions which satisfy the postulates of § 2 have the mean-value property, that is, if x_0, y_0 is the center of a circle C lying entirely inside of R, the region for $f(x, y)$, any function of the class, then $f(x_0, y_0)$, its value at the center of C is equal to the average value of $f(x, y)$ taken over the circumference of C.

To show this, we first form a series of positive constants ε_i such that $\lim_{i\to\infty} \varepsilon_i = 0$. We select a particular ε_i, and divide the circumference of the circle C into N_i equal parts, taking N_i so large that the oscillation of $f(x, y)$ on any one part is less than ε_i. This is possible since postulate 1 makes $f(x, y)$ continuous, and therefore uniformly continuous on the circumference of C. We next form $N_i - 1$ new functions, existing in and on C, obtained from $f(x, y)$ by rotating the axes about the center of C through an angle $2k\pi/N_i$ $(0 < k < N_i)$. These functions, by postulate 3, are members of the class, as is also the function $f_i(x, y)$, defined as the sum of these $N_i - 1$ functions and the original function, divided by N_i, by postulate 2. Finally, from postulates 2 and 4, we see that the function $F_i(x, y) = f_i(x, y) - f_c$, where f_c denotes the average value of the original function on the boundary of C, is also in the class.

The function $F_i(x, y)$ satisfies the inequality

$$|F_i(x_c, y_c)| < \varepsilon_i, \tag{1}$$

where x_c, y_c denote any point on C. For, if p_k is the average value of $f(x, y)$ in the kth subdivision of C, we have

$$f_c = \sum_{k=1}^{N_i} p_k/N_i, \tag{2}$$

and from the choice of N_i, for x_c, y_c in the kth subdivision

$$|f(x_c, y_c) - p_k| < \varepsilon_i. \tag{3}$$

Recalling the method of forming $f_i(x, y)$, we deduce from (2) and (3) that

$$|f_i(x_c, y_c) - f_c| < \varepsilon_i, \tag{4}$$

for all points on the circumference of C, and this is equivalent to (1) above.

Moreover, at x_0, y_0, the center of the circle C, we have

(5) $$F_i(x_0, y_0) = f_i(x_0, y_0) - f_c = f(x_0, y_0) - f_c.$$

Thus the values of the functions $F_i(x, y)$ at x_0, y_0 are the same for all values of i, and consequently approach a limit when i becomes infinite. But from (1) and postulate 5, this limit must be zero, and (5) gives

(6) $$f(x_0, y_0) = f_c,$$

which is the mean-value property.

4. *The Harmonic Character of the Functions.* We may now readily prove that the functions of our class are harmonic at all interior points of their regions R. For, with any interior point of a region R as center, let us draw a circle lying wholly inside R, the region for the function $f(x, y)$. We also form, by Poisson's integral or otherwise, a harmonic function $h(x, y)$ having the same values along the circumference of the circle as the given function $f(x, y)$. Since both $h(x, y)$ and $f(x, y)$ possess the mean-value property, their difference $f(x, y) - h(x, y)$ will also possess it. Consequently this function takes on its maximum and minimum values on the circumference of the circle; and since it is zero on the circle, it must be zero identically. Hence $f(x, y)$ agrees with $h(x, y)$ in a neighborhood of the center of the circle and is harmonic at this point.

5. *Postulates for Three Dimensions.* It is fairly obvious what revisions we must make in the postulates of § 2 to make them applicable to potential functions in three-dimensional space. Here we deal with a class of functions of three variables, or Cartesian coordinates, x, y and z, where each function has associated with it a three-dimensional region R, of space. Our assumptions now are:

(1) Each function is continuous in all three variables at all interior points of its region R.

(2) If R_1 and R_2, the regions for two functions of the class $f_1(x, y, z)$ and $f_2(x, y, z)$, have a region R_3 in common, then any linear combination of these functions, such as $Af_1(x, y, z) + Bf_2(x, y, z)$, is a function of the class, whose region contains all the points of R_3.

(3) If an orthogonal transformation of the variables (i. e., a change of Cartesian axes) converts the function $f(x, y, z)$ with region R into $F(x', y', z')$, this latter function is a member of the class, its region being the region R expressed in the new variables.

(4) The function $f(x, y) = 1$ is a function of the class, its region R being the whole of space.

(5) If a given sphere lies entirely inside each region R_i, belonging to $f_i(x, y, z)$, these $f_i(x, y, z)$ constituting an infinite sequence of functions of the class, and if, for points x_s, y_s, z_s on the surface of the sphere $\lim_{i \to \infty} f_i(x_s, y_s, z_s) = 0$, the limit being approached uniformly over the surface; then the sequence of values of the functions $f_i(x, y, z)$ at the center of the sphere cannot approach a limit different from zero.

Any class of functions satisfying these five postulates is, as we shall show presently, a class of Newtonian potential functions. Consequently the widest class of functions satisfying them is the totality of Newtonian potential functions. The remarks made about our earlier set of postulates at the end of § 2 apply here, mutatis mutandis.

Our reason for writing out the postulates at length and carrying out the proof of their sufficiency for the three-dimensional case, is that the transition to the three-dimensional case requires an essential modification in the proof. For, in the earlier proof of § 3, use was made of a series of regular polygons approximating a circle. As no analogous configuration exists in space, we must resort to a property of functions on a sphere, to which we next proceed.

6. *Functions on a Sphere.* If we are given a function on a sphere, we may form a new function from it by the following process. We select a particular diameter of the sphere as an *axis*, and so determine the second function that its value is constant on every circle on the sphere whose plane is perpendicular to the axis, and is equal to the average of the original function on this same circle. We define this process as that of *averaging the original*

function about an axis. We may now state the following lemma.

LEMMA. *If a continuous function on a sphere is averaged about one axis, the resulting function about a second axis perpendicular to the first one, this third function about the first axis, and so on, indefinitely, using the two axes alternately, the sequence of functions so obtained will approach a limit. Furthermore, this limit will be constant over the whole sphere, and equal to the average of the original function over the sphere.*

To prove this, we first observe from the continuity of the function that it possesses a maximum and a minimum value, which are actually reached, since the surface of the sphere is closed. Also, from the nature of the averaging process, the successive functions are all continuous, and the maximum values M_i form a never increasing sequence, while the minimum values m_i form a never decreasing one. Hence these quantities approach limits, which we denote by M and m respectively.

Another consequence of the character of the averaging process is that if we determine a positive δ such that the oscillation of our original function is less than a preassigned ϵ in every circle on the sphere of radius δ, which we may do since this function is continuous and therefore uniformly continuous on the sphere, the same δ will retain its relation to ϵ after the averaging, and hence for all subsequent functions in the series. This follows from the fact that the oscillation in a circle of radius δ after averaging cannot exceed the maximum oscillation in the set of circles of equal radius having their centers in the plane perpendicular to the axis containing the center of the original circle.

Let us now select an ϵ and determine a δ in the way described above, and let us set

$$\eta = \frac{2\epsilon\delta}{\pi r}, \tag{7}$$

where r is the radius of the sphere. We next consider the result of averaging the original function i times, taking i so

great that F_i, the resulting function, has its maximum and minimum values each within η of their respective limits, i. e.,

$$\begin{cases} M \leqq M_i \leqq M+\eta, \\ m-\eta \leqq m_i \leqq m. \end{cases} \tag{8}$$

Let E be the constant value of the function F_i on the equator, i. e., on the great circle perpendicular to the axis about which we have just averaged, and consequently having as one diameter the other axis which we must use to get F_{i+1}. From the way we selected δ, F_i cannot differ from E by more than ε in the zone of width 2δ bounded by small circles parallel to and at distances δ above and below the equator. Hence, if we construct two lunes lying entirely in this zone, each of angular width $2\delta/r$, bounded by planes through the new axis, the value of F_i will be at most $E+\varepsilon$ inside these lunes. The remaining portion of the sphere consists of two lunes each of angular width $\pi-2\delta/r$, and in these lunes, the function is at most M_i or $M+\eta$ by (8). Consequently, if we average once more to get F_{i+1}, since this function cannot exceed the average of the upper bounds in the lunes in question, we must have

$$F_{i+1} \leqq \frac{\frac{4\delta}{r}(E+\varepsilon)+\left(2\pi-\frac{4\delta}{r}\right)(M+\eta)}{2\pi}. \tag{9}$$

Since F_{i+1} is at least as great as M at some point, this implies that

$$M \leqq M+\eta+\frac{2\delta}{\pi r}(E+\varepsilon-M-\eta), \tag{10}$$

from which, using (7), we get

$$E \geqq M+\eta-2\varepsilon \geqq M-2\varepsilon. \tag{11}$$

By an entirely analogous argument, using the minimum values of F_i and F_{i+1}, we show that

$$E \leqq m+2\varepsilon. \tag{12}$$

These last two equations give

$$E+2\varepsilon \geqq M \geqq m \geqq E-2\varepsilon, \tag{13}$$

which shows that $M=m$, since ε can be made arbitrarily small.

Hence the value of the function at all points approaches this constant value, and since the averaging process leaves the average of the function taken over the entire sphere unchanged, the constant must be equal to this average, or the average of the original function over the sphere.

7. *Deduction of the Mean-value Property.* We are now in a position to prove that the class of functions which satisfy the postulates of § 5 have the mean-value property. That is, if x_0, y_0, z_0 is the center of a sphere S lying entirely inside of R, the region for $f(x, y, z)$ any function of the class, then $f(x_0, y_0, z_0)$, its value at the center of S, is equal to the average value of $f(x, y, z)$ taken over the surface of S.

Since the process of averaging about an axis discussed in the preceding section involves limiting processes not covered by our postulates, we cannot directly apply it; but we can obtain a function of our class from $f(x, y, z)$ having the same value at the center of S, and on S approximating the result of averaging $f(x, y, z)$ about any axis as closely as we please. For we have merely to select an N so large that the oscillation of the original function in any circle of radius $\pi r/N$ is less than ϵ, form functions obtained from the original function $f(x, y, z)$ by rotating the coordinates about the chosen axis through angles $2k\pi/N$ $(0 < k < N)$, sum these functions and divide by N. The proof that this is the desired function is entirely parallel to the argument given in § 3.

Let us now form a series of positive constants ϵ_i such that $\lim_{i \to \infty} \epsilon_i = 0$. For each ϵ_i we shall form a function $f_i(x, y, z)$ as follows. First form a function $f_{i1}(x, y, z)$ which has the same value at the center of S as $f(x, y, z)$ and which, on the surface of S, differs from the result of averaging $f(x, y, z)$ about a fixed axis by less than $\epsilon_i/4$, which can be done by the method just described. Next select a second axis perpendicular to the first, and form $f_{i2}(x, y, z)$ which, on the surface of S, differs from the result of averaging $f_{i1}(x, y, z)$ about this second axis by

less than $\epsilon_i/8$. Then form $f_{i3}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i2}(x, y, z)$ about the first axis by less than $\epsilon_i/16$ and so on; $f_{it}(x, y, z)$ differing on the surface of S from the result of averaging $f_{i,t-1}(x, y, z)$ about the first or second axis, according as t is odd or even, by less than $\epsilon_i/2^{t+1}$. The functions on the surface of S, $F_1(x, y, z), \ldots, F_t(x, y, z), \ldots$ obtained from $f(x, y, z)$ by actually averaging about the two axes in succession, will be related to the set just constructed, in that, on the surface of S,

$$|F_1 - f_{i1}| < \epsilon_i/4, \qquad |F_2 - f_{i2}| < \epsilon_i/2^2 + \epsilon_i/2^3, \tag{14}$$

and for all values of t,

$$|F_t - f_{i1}| < \epsilon_i(1/2^2 + 1/2^3 + \cdots + 1/2^{t+1}) < \epsilon_i/2. \tag{15}$$

But, by the lemma of § 6, the limit of $F_t(x, y, z)$ as t becomes infinite is f_s, the average of $f(x, y, z)$ over the surface of S. Hence, by taking t large enough, we may make

$$|F_t - f_s| < \epsilon_i/2; \qquad t \geqq T. \tag{16}$$

We put

$$f_i(x, y, z) = f_{iT}(x, y, z). \tag{17}$$

Then, from (15) and (16), we have

$$|f_i(x, y, z) - f_s| < \epsilon_i. \tag{18}$$

Furthermore, $f_i(x_0, y_0, z_0)$ is equal to $f(x_0, y_0, z_0)$, since all the processes used to obtain $f_i(x, y, z)$ from $f(x, y, z)$ left its value at the center of S unchanged. Thus the sequence of functions

$$G_i(x, y, z) = f_i(x, y, z) - f_s \tag{19}$$

satisfies all the conditions of our fifth postulate. Since its value at the center of S is constant, this constant must be zero, and we have

$$f(x_0, y_0, z_0) = f_i(x_0, y_0, z_0) = f_s \tag{20}$$

which is the mean-value property.

8. *The Harmonic Character of the Functions.* We may now apply the reasoning of § 4 to show that inside any sphere S entirely within its region R, any function of our

class $f(x, y, z)$ is identical with the harmonic function set up by Poisson's integral which has the same values on the boundary of S as $f(x, y, z)$. Thus all the functions of our class are harmonic at all points interior to their regions. Also since every linear class of harmonic functions in three variables satisfies them, the widest such class of functions is the totality of Newtonian potential functions.

9. *Concluding Remarks.* We might extend our postulates to the case of n dimensions, the proof being accomplished by a proper generalization of the lemma of § 6.

It should also be noticed that the postulates for two dimensions apply to functions on a sphere* (and for three dimensions to functions on a hypersphere) provided we replace the orthogonal transformations which correspond to a rotation of the plane (or 3-space) by the transformations on the sphere (hypersphere) corresponding to rotations about a diameter, as the proof given still applies with but slight modifications.

Moreover, since postulate 5 is a consequence of the maximum-minimum property, which is, of course, much more restrictive, we may obtain a set of postulates, by postulating the maximum-minimum property, which, though more stringent than ours, is stated in terms of more familiar properties. Since this substitution enables us to dispense with postulate 4, the new assumptions may be stated in the following form:

A linear family of functions of two (three) variables, invariant as a family under orthogonal transformations of the variables, each function being continuous at all interior points of some region R, and such that for any subregion of R it takes its maximum and minimum values on the boundary of this region, is necessarily a family of harmonic functions.

This statement is of interest, because it shows one set of conditions which, on being added to the maximum-minimum property, characterize the class as harmonic.

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* This extension was called to my attention by Dr. Norbert Wiener.

ON THE LOCATION OF THE ROOTS OF POLYNOMIALS*

BY J. L. WALSH

It is the object of this note to prove a number of results (particularly Theorems I, II, V, VI, below) concerning the roots of polynomials, generalizations of former results established by the writer. Our main new result is the following theorem.

THEOREM I. *Let (the interiors and boundaries of) the circles $C_1, C_2, \ldots, C_k$ whose centers are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the respective loci of $n_1, n_2, \ldots, n_k$ roots of a variable polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If the polynomial*

$$(1)\quad a_0 z^n + n a_1 z^{n-1} + n(n-1) a_2 z^{n-2} + \cdots + n(n-1)\ldots 2\cdot 1\, a_n, \qquad (n = n_1 + n_2 + \cdots + n_k),$$

has only real roots, and if the circles C_i are sufficiently small, then the locus of the roots of the polynomial

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

consists of the circles C_j' which are equal to the circles C_i and whose centers are the roots of $F(z)$ when the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ of multiplicities $n_1, n_2, \ldots, n_k$ respectively.† Any circle C_j' which has no point in common with any of the other circles C_m' contains a number of roots of $F(z)$ equal to the multiplicity of its center as a root of $F(z)$ when the roots of $f(z)$ are the points α_i.

We shall later make clear the exact meaning of the words, "if the circles C_i are sufficiently small". Theorem I is to be proved by iteration of the following theorem.

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† When the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$, the roots of $F(z)$ are all collinear with the roots of $f(z)$. This is a well known theorem due to Hermite, which can easily be established by proving the result in succession for the polynomials $F_1(z), F_2(z), \ldots, F_n(z)$ used below.

THEOREM II. *Let the circles $C_1, C_2, \ldots C_k$ whose centers are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the loci respectively of $n_1, n_2, \ldots, n_k$ roots of a variable polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If c_1 is real, then the locus of the roots of the polynomial*

$$F_1(z) = f(z) + c_1 f'(z) \tag{2}$$

consists of the circles C_j' which are equal to the circles C_i and whose centers are the roots of $F_1(z)$ when the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ of multiplicities $n_1, n_2, \ldots, n_k$ respectively. Any circle C_j' which has no point in common with any other of the circles C_m'' contains a number of the roots of $F_1(z)$ equal to the multiplicity of its center as a root of $F_1(z)$ when the roots of $f(z)$ are the points α_i.

Theorem I has already been established for the case

$$a_0 = a_1 = \cdots = a_{e-1} = a_{e+1} = a_{e+2} = \cdots = a_n = 0, \quad a_e \neq 0.^*$$

Theorem I has also been proved with no restriction on (1) nor on the size of the circles C_i for the case $k = 1$.† The limiting case for $c_1 = \infty$ of Theorem II is also true, and is proved as a limiting case of I, Theorem VIII. For the case that the circles C_i of Theorem II are sufficiently small, Theorem II is contained in Theorem I.

The limiting case for $c_1 = \infty$ of Theorem II gives

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* TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 31–69, Theorem X, p. 53. We shall refer to this paper as I.

† TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 163–180, Theorem VI. We shall refer to this paper as II.

The note in the COMPTES RENDUS of which II is the development was published and II itself was written before the publication of a paper by S. Kakeya, PROCEEDINGS OF THE PHYSICO-MATHEMATICAL SOCIETY OF JAPAN, (3), vol. 3 (1921), pp. 94–100. Kakeya treats the main theorem of II, and by essentially the methods of II, although he makes the restriction that the circular region C involved shall be bounded by a circle whose center is the origin. See also T. Takagi, PROCEEDINGS OF THE PHYSICO-MATHEMATICAL SOCIETY OF JAPAN, (3), vol. 3 (1921), pp. 175–179, who gives a proof of Theorem V of the present paper.

a theorem which is essentially a well known theorem due to Lucas:

If a circle contains all the roots of a polynomial, that circle contains all the roots of its derivative.

Theorem I contains the more general theorem of Lucas which deals with any derivative of the original polynomial, for it turns out that in this case the circles C_i are "sufficiently small" in the sense which we shall give to those words.

Let us proceed to the proof of Theorem II. In the case $c_1 = 0$, the hypothesis and conclusion are the same; we turn to the case $c_1 \neq 0$. The roots of $F_1(z)$ are the roots of

$$(3) \quad -\frac{1}{c_1} = \frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n},$$

where $z_1, z_2, \ldots, z_n$ are the roots of $f(z)$. It follows from the form of (3) that we can interpret the conjugate imaginary complex quantity of the right-hand member of (3) as the force due to particles situated at the respective points $z_1, z_2, \ldots, z_n$, each of which repels with a force equal to the reciprocal of the distance. The roots of (3) are the positions of equilibrium in the field of force due to these particles and an additional constant force of magnitude $1/c_1$ at every point of the plane. It is to be noted that a multiple root of $f(z)$ is always a root of $F_1(z)$ although not a root of (3).

The proof of Theorem II can be given so as to be almost identical with the proof of the limiting case for $c_1 = \infty$ of Theorem II.* That is, we consider one particular root z of $F_1(z)$, and for that point z we replace the n repelling particles of (3) by n equivalent particles which coincide at a point ζ. When the n particles have the circles C_i as their respective loci, the n-fold particle ζ has a

* This remark applies not to the proof of I, Theorem X for $k = 1$ as actually given in I, but to the proof there given in detail for I, Theorem VI. That proof can of course be given so as to apply directly to I, Theorem X, for $k = 1$.

circular region as its locus. Study of the locus of ζ will give us the theorem. Let us proceed to the general outline of the proof. We shall omit details if they are similar to the details given in I.

Theorem II is surely true when there is only one circle C_i. In this case, if a point z is on the boundary of the locus of the roots of $F_1(z)$ and is not on or within the circle C_1 which contains more than one particle, then it is necessary that

$$z_1 = z_2 = \cdots = z_n,$$

and these points lie on C_1.* Thus the line zz_1 is parallel to the axis of reals.

If the locus of all the roots of $f(z)$ is the *exterior* of a circle C_1, the locus of the roots of $F_1(z)$ consists of the exterior of C_1 (if $n > 1$), and the exterior of another circle C_1' defined as the circle traced by that root of $F_1(z)$ other than z_1 when the point $z_1 = z_2 = \cdots = z_n$ traces the circle C_1. If z is a root of $F_1(z)$ interior to the circle C_1 and on the boundary of its locus, z must lie on C_1', the point $z_1 = z_2 = \cdots = z_n$ lies on C_1, and the line zz_1 is parallel to the axis of reals.

We return to Theorem II for the polynomial $F_1(z)$, and suppose that there are precisely two circles C_i, namely C_1 and C_2. It is clear from the field of force that no point z not lying on or between the common external tangents to C_1 and C_2 can be a root of (3). Moreover it follows from the special case of Theorem I mentioned above that not every point z lying between these tangents can be a point of the locus of the roots of (3). In the general case $n_1 > 1$, $n_2 > 1$, there are four circles C_j' of which two are the circles C_1 and C_2, and one of which has its center interior to the interval α_1, α_2.

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* Here and below we make use of the lemma: *The force at a point P due to k particles situated in a circular region not containing P is equivalent to the force at P due to k coincident particles also in C.* See Walsh, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 101–116, Lemma I, p. 102.

Let z be a fixed point on the boundary of the locus of the roots of $F_1(z)$. Then z is not interior to a circle C_1 or C_2 which is the locus of more than one root of $f(z)$, and is not interior to both circles, for in either of these cases z as well as any neighboring point can be made a multiple root of $f(z)$ and hence a root of $F_1(z)$. Suppose also that z is not on C_1 if $n_1 > 1$ nor on C_2 if $n_2 > 1$. Then (as in I, p. 47) we may replace the n_1 particles of C_1 by n_1 equivalent but coincident particles ζ_1 in C_1, the n_2 particles of C_2 by n_2 equivalent but coincident particles ζ_2 in C_2; and finally we may replace these particles ζ_1 and ζ_2 by an equivalent n-fold particle ζ. When ζ_1 and ζ_2 have the circles C_1 and C_2 as their loci, the locus of ζ will be either the interior of a circle C or the exterior of a circle C, and C will cut the line zz' through z parallel to the axis of reals either at the same angle as do the circles C_1 and C_2 or at the supplementary angle according as the locus of ζ is interior or exterior to C.*

Corresponding to the particular point z that we are considering on the boundary of its locus, we may choose definite points $z_1, z_2, \ldots, z_n, \zeta_1, \zeta_2, \zeta$, so that z is a position of equilibrium in the field of force. The point ζ must lie on the boundary of its locus C, and z must lie on the boundary of the locus of the roots of $F_1(z)$ when the locus of ζ $(= z_1 = z_2 = \cdots = z_n)$ is considered merely to be the locus C. For a small change in z makes only a small change in C. If z, ζ, and C have not the relation stated, a sufficiently small but arbitrary change of z can be made, and ζ can be chosen in the new locus C so that z is still a root of $F_1(z)$. That is, the original point z is not on the boundary of its locus.

The point ζ must therefore lie at an intersection of the

* It is conceivable that this locus of ζ should consist of the entire plane. This slight difficulty can be overcome as in I, p. 50. It is possible that the locus of ζ should be a half-plane. The reader will make the necessary modifications in the present treatment, which are indeed modifications of phraseology rather than of method.

line zz' and the circle C; ζ is on the boundary of its locus. By the properties of the circles C_1, C_2, C, and the angles in which they are cut by the line zz' (as in I, p. 49), the points ζ_1 and ζ_2 must lie on C_1 and C_2 respectively and on the line zz'. In fact, ζ_1 and ζ_2 must be those intersections of zz' with C_1 and C_2 which cause z, the root of the corresponding polynomial $F_1(z)$, to lie on one of the circles C_j'. Thus we have proved that whenever z lies on the boundary of its locus, it lies on one of the circles C_j'.

The locus of the roots of $F_1(z)$ contains all the points interior to the circles C_j'; indeed, it is obvious by a simultaneous translation of the points α_i as the roots of $f(z)$ and of the roots of the corresponding polynomial $F_1(z)$ that every point on or within a circle C_j' is a point of the locus. Every point of the boundary of the locus is a point of one of the circles C_j', no point not between the two common external tangents of the circles C_i, C_j' is a point of the locus, so the locus consists of precisely the points stated in Theorem II.

The remark in Theorem II concerning the number of roots of $F_1(z)$ in the regions C_j' follows (as in I, p. 50) from a consideration of the roots of $F_1(z)$ when the roots of $f(z)$ are the points α_i, and from the continuity of the roots of $F_1(z)$ considered as functions of the roots of $f(z)$. Theorem II is now completely proved for the case $k = 2$.

The same method is used in every case. We replace the particles $z_1, z_2, \ldots, z_n$ by a single equivalent particle ζ, whose locus is a circular region. This particle ζ is on the boundary of its locus if z is on the boundary of its locus, and the properties of the boundary of the locus of ζ in connection with the properties of the circles C_i enable us to prove that the point z is on one of the circles C_j', and to prove the theorem in its generality. Further details are left to the reader.

Theorem II is a result dealing with the interiors of circles as the loci of roots of polynomials. We may use the same method to prove a result similarly dealing with the *exteriors* of equal circles whose centers lie on a line parallel to the

axis of reals, or dealing with half-planes bounded by lines perpendicular to the axis of reals. Rotation of the plane in any of these cases or in Theorem I gives a new result.

We proceed to prove Theorem I by successive application of Theorem II. Suppose $a_0 \neq 0$ and denote by $c_1, c_2, \ldots, c_n$ the negatives of the various roots of (1); these have been assumed real. Theorem I has been proved for the polynomial

$$F_1(z) = f(z) + c_1 f'(z). \tag{4}$$

From this fact follows Theorem I for the polynomial

$$F_2(z) = F_1(z) + c_2 F_1'(z) = f(z) + (c_1 + c_2) f'(z) + c_1 c_2 f''(z).$$

For, by the part of the theorem already proved and applied to $F_1(z)$, the locus of the roots of $F_1(z)$ consists of certain circles. Since the roots of $F_1(z)$ lie in these circles, it follows that the roots of $F_2(z)$ lie in certain other circles, which are precisely the circles C_j' of Theorem I that pertain to $F_2(z)$ as a linear combination of $f(z)$ and its derivatives. It follows from a simultaneous translation of the α_i and the roots of $f(z)$ that every point of the last-mentioned circles is a point of the locus of the roots of $F_2(z)$. Thus Theorem I is proved for $F_2(z)$ and by induction can be proved for

$$\begin{cases} F_3(z) = F_2(z) + c_3 F_2'(z), \\ F_4(z) = F_3(z) + c_4 F_3'(z), \\ \cdots\cdots\cdots, \\ F_n(z) = F_{n-1}(z) + c_n F_{n-1}'(z) = F(z). \end{cases} \tag{5}$$

It should be noted, however, (as in I, p. 53), that the reasoning just used is not of universal validity. For if the circles C_j' which contain the roots of $F_1(z)$ are not mutually external, we cannot say that they contain respectively the proper number of roots of $F_1(z)$ for a new application of Theorem II.* Our reasoning is valid only in the case that the circles C_j' which are the loci of the roots of $F_1(z)$ have no point in common one with another, and similarly for the sets of circles C_j' which are the loci of the roots of

* This is, in fact, not merely a fault of our particular statement or method of proof of Theorem II. See I, pp. 36, 37.

$F_2(z)$, $F_3(z)$, ..., $F_{n-1}(z)$. This condition will always be satisfied in Theorem I if, when the polynomial (1) and the points α_i are given, the common radius of the circles C_i is sufficiently small, and this is the meaning we give to the restriction in the enunciation of the theorem, "if the circles C_i are sufficiently small". In order to prove the theorem up to but not including the last sentence, we need not require that no two of the circles C_j' which are the loci of the roots of $F_n(z) = F(z)$ should have a point in common, but if this condition is satisfied the number of roots of $F(z)$ in those circles C_j' is as indicated. This fact is proved (as in I, p. 50) from the continuity of the roots of $F(z)$ considered as functions of the roots of $f(z)$.

It still remains, in the proof of Theorem I, to remove the restriction $a_0 \neq 0$. This can be accomplished by remembering that Theorem I has already been proved* when $F(z)$ is simply a derivative of $f(z)$. Hence application of Theorem I for the case $a_0 \neq 0$ to a derivative of $f(z)$ shows that the roots of $F(z)$ must lie in the circles C_j'. Every point on or within the circles C_j' is a point of the locus, for this follows by a translation. We naturally require as before that the circles C_i be sufficiently small. The number of roots of $F(z)$ in a circle C_j' can be determined by continuity, so Theorem I is completely established.

We add the statement of a theorem which is almost trivial when the preceding development is considered, but which seems nowhere to have been mentioned in the literature.†

THEOREM III. *If the roots of a polynomial $f(z)$ of degree n lie in a region R bounded by two parallels to the axis of reals, and if the roots of* (1) *are all real, then the roots of*

* In I, Theorem X. The proof of the present note is also valid for this case; we merely choose the left-hand member of (3) to be zero to prove the result for the first derivative, and apply repeatedly for the other derivatives.

† In connection with these theorems for the polynomial $F_1(z)$, see Fujiwara, TÔHOKU MATHEMATICAL JOURNAL, vol. 9 (1916), pp. 102–108. See also Uchida, TÔHOKU MATHEMATICAL JOURNAL, vol. 10 (1916), pp. 139–141.

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

also lie in R.

The generalization of Theorem I for the case $k = 1$ but which makes no restriction on (1) nor on the size of the circles C_i (i. e., II, Theorem VI) gives us the following theorem.

THEOREM IV. *If all the roots of a polynomial $f(z)$ lie in the strip*

$$a_1 \leqq y \leqq a_2$$

of the complex $(z = x + iy)$ plane, and if all the roots of the polynomial (1) *lie in the strip*

$$b_1 \leqq y \leqq b_2,$$

then all the roots of

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

lie in the strip

$$a_1 + b_1 \leqq y \leqq a_2 + b_2.$$

Innumerable theorems of the same nature as Theorem IV can be proved by II, Theorem VI. We give one further example, after a preliminary remark.

If two convex regions R_1 and R_2 are the respective loci of points z_1 and z_2, then the region R which is the locus of the point $z = z_1 + z_2$ is also a convex region. We shall speak of R as the *sum* of the regions R_1 and R_2. Then we have

THEOREM V. *If two convex regions are the respective loci of the roots of $f(z)$ and of* (1)*, then their sum is the locus of the roots of $F(z)$.*

Theorem V obviously includes Lucas's theorem for convex regions on the roots of the derivative of a polynomial. Theorem V includes, and can be proved from Theorem IV. Theorem IV includes Theorem III, and Theorem III includes the theorem due to Hermite which we have been using in Theorem I, to the effect that if the roots of $f(z)$ and of (1) are real, then the roots of $F(z)$ are also real.*

* Reference has been made Takagi's work in this connection. Theorem V does not directly extend to the case of infinite convex

Let us study the case of real polynomials in more detail. We shall prove for them a theorem analogous to Theorem I.

THEOREM VI. *Let intervals I_i $(i = 1, 2, \ldots, k)$ of the axis of reals, whose extremities are α_i, β_i $(\alpha_i \leqq \beta_i)$ be the respective loci of n_i roots of a variable polynomial $f(z)$ which has no other roots. Then if the roots of the polynomial* (1) *are all real, the locus of the roots of $F(z)$ is composed of a number of intervals I_j' of the axis of reals. The left-hand extremities of the intervals I_j' are the roots of $F(z)$ when the roots of $f(z)$ are concentrated at the points α_i; the right-hand extremities are the corresponding roots of $F(z)$ when the roots of $f(z)$ are concentrated at the points β_i. Any interval I_j' which has no point in common with any other interval I_j' contains a number of roots of $F(z)$ equal to the multiplicity of its left-hand extremity as a root of $F(z)$ when the roots of $f(z)$ are the points α_i. If the intervals I_i are all of the same length, the intervals I_j' are also all of this same length.*

Theorem VI has been proved (II, footnote, p. 180) for the case $k = 1$, and has been proved (I, Theorem XV) for general k for the case that $F(z)$ is a derivative of $f(z)$. The proof of this latter theorem is to be followed closely in the proof of Theorem VI. We shall prove Theorem VI under the assumption that none of the intervals I_i is a point; to include this more general case requires merely a slight change in phraseology. In the theorem as stated the intervals are assumed to be finite, but the theorem can be extended to include infinite intervals. We prove the theorem first for the case of the polynomial

$$F_1(z) = f(z) + c_1 f'(z).$$

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regions, using the ordinary extension of the definition of convexity, although the following is an immediate result of II, Theorem VI:

If the roots of (1) *have as their locus any region R and if the roots of $f(z)$ have as their locus the region C,*

$$|z - a| \leqq r, \text{ where } a \text{ and } r \text{ are fixed,}$$

then the locus of the roots of $F(z)$ is the region S which is the locus of points

$$\zeta = \rho + z,$$

where ρ and z have the respective loci R and C. If R is the interior or exterior of a circle, so also is S.

Let us denote by $\alpha_i^{(m)}$ the roots of $F_m(z)$ when the roots of $f(z)$ are concentrated at the points α_i, $\alpha_i^{(m)} \leqq \alpha_j^{(m)}$ when $i<j$, and similarly by $\beta_i^{(m)}$ the roots of $F_m(z)$ when the roots of $f(z)$ are concentrated at the points β_i, $\beta_i^{(m)} \leqq \beta_j^{(m)}$ when $i<j$. We shall show that the intervals $(\alpha_i^{(m)}, \beta_i^{(m)})$ form the locus of the roots of $F_m(z)$.

We start with the roots of $f(z)$ concentrated at the points α_i and move these roots continuously toward the right until they reach the points β_i. The roots of $F_1(z)$ also vary continuously; they start at the points α_i' and reach the points β_i'. We can even say that the rth root z_r' of $F_1(z)$ (roots numbered in order from the left) varies continuously. Let us now prove that z_r' always moves, if at all, toward the right.

The equation determining z_r' is of the form

$$\Phi = \frac{1}{c_1} + \frac{f'(z)}{f(z)} = \frac{1}{c_1} + \frac{m_1}{z_r' - \gamma_1} + \frac{m_2}{z_r' - \gamma_2} + \cdots + \frac{m_s}{z_r' - \gamma_s}, \tag{6}$$

where the γ_i are the roots of $f(z)$, coinciding in any multiplicities desired. We compute the values

$$\frac{\partial \Phi}{\partial z_r'} = -\frac{m_1}{(z_r' - \gamma_1)^2} - \frac{m_2}{(z_r' - \gamma_2)^2} - \cdots - \frac{m_s}{(z_r' - \gamma_s)^2},$$

$$\frac{\partial \Phi}{\partial \gamma_i} = \frac{m_i}{(z_n' - \gamma_i)^2}.$$

It is always true that $\partial z_r'/\partial \gamma_i$ is positive, so z_r' always increases with γ_i.*

* This result is the chief tool in the proof of Theorem VI, and as it is true that $\partial z_r'/\partial \gamma_i \neq 0$ for a large class of polynomials, Theorem VI can be suitably modified to apply to much more general polynomials, in particular to $f'(z)$, where $f(z)$ is real but has not necessarily all real roots. Theorem VI holds without change for polynomials and the iterates of polynomials of the type

$$(a - b^2 z) f(z) + c f'(z),$$

where a, b, c are real and the roots of $f(z)$ are real. This last class of polynomials has recently been considered by Nagy, JAHRESBERICHT DER VEREINIGUNG, vol. 31 (1922), pp. 238–251.

Equation (6) is no longer valid to determine z_r' if z_r' is located at a multiple root of $f(z)$. Under these circumstances, if γ_i does not coincide with z_r', the motion of γ_i does not change the position of z_r'. If γ_i does coincide with z_r' and if γ_i is moved to the right, z_r' is either unchanged or moved to the right; this follows immediately from the fact that a t-fold root of $f(z)$ is a $(t-1)$-fold root of $F_1(z)$, and from the fact that every interval of the axis of reals bounded by roots of $f(z)$ contains at least one root of $F_1(z)$.

From the general fact, then, that the rth root z_r' of $F_1(z)$ varies continuously and in one sense under the indicated variation of the roots of $f(z)$, it follows that z_r' traces the entire interval from α_r' to β_r' and can never lie outside that interval. The determination of the locus of Theorem VI is now complete for the polynomial $F_1(z)$. The number of roots of $F_1(z)$ in an interval (α_r', β_r') having no point in common with any other interval (α_s', β_s') is easily found by continuity.

When the roots of $f(z)$ move from the points α_i continuously to the right, the roots of $F_1(z)$ move from the points α_i' continuously to the right, and from the formulas (5), the roots of $F_2(z)$ move from the points α_i'' continuously to the right, and the roots of $F_m(z)$ move from the points $\alpha_i^{(m)}$ continuously and to the right. This gives us Theorem VI for the case $a_0 \neq 0$; the restriction $a_0 \neq 0$ can be removed with no difficulty as in the proof Theorem I of the present paper.

HARVARD UNIVERSITY

NATIONAL RESEARCH COUNCIL REPORT ON ALGEBRAIC NUMBERS

Algebraic Numbers. Report of the Committee on Algebraic Numbers, National Research Council. By L. E. Dickson, H. H. Mitchell, H. S. Vandiver, G. E. Wahlin. BULLETIN OF THE NATIONAL RESEARCH COUNCIL, vol. 5, part 3, number 28, February, 1923. Washington. D. C. 96 pp.

Nowhere is the craft and subtlety of the mathematician more ingeniously developed than in the theory of algebraic numbers. The subject is dotted over with gins and snares, not to say boobytraps, and more than one reputable arithmetician has fallen foul of its pitfalls, thereby exhibiting himself as a warning to the initiated who have sense enough to heed his misfortunes. The unwary, with all the courage of their kind, rush in and are quickly slain. But for those who are gifted with caution and perseverance the theory has rich rewards. As an emancipator of the intellect the theory of ideals is at least the peer of non-euclidean geometry, yet it is but little cultivated by professional mathematicians and is almost wholly unappreciated by mathematical philosophers. The reason is not far to seek: even schoolboys are easily if superficially impressed by the spectacular heterodoxies of geometry, while hard labor and a mature judgment are prerequisites for the mere apperception of the quieter and perhaps more profound beauties of modern arithmetic. Should some mathematician with the necessary talents popularize this subject as non-euclidean geometry has been popularized, he would do an important service to both mathematics and philosophy.

The Report by Dickson, Mitchell, Vandiver, and Wahlin has the twofold object of bringing up to date Hilbert's classic *Bericht* of 1894–95 on the theory of algebraic number fields, and of supplementing the Bericht by accounts of the literature on fields of functions and related topics which Hilbert omitted. It is stated in the preface that Hilbert's report and the present one exhaust the literature. So far as the reviewer can judge this claim is substantiated. There are a few printer's errors, but none that will cause any inconvenience.

This Report differs in one essential respect from both Hilbert's Bericht and from the Bulletins on mathematical physics issued by the National Research Council. It would not be possible to gain a working knowledge of the topics discussed from the extremely condensed statements in the 96 pages of the Report. Nevertheless this Report is admirably fitted to the use of the experts for whom it doubtless is intended.

The responsibility for the several main divisions of the Report is as follows: quadratic, galois and abelian fields, units in a general field,

(MITCHELL); cubic fields, norms and congruences in a general algebraic field, classes of ideals, Klassenkörper, complex multiplication of elliptic functions, higher reciprocity laws and Kummer fields, (VANDIVER); distribution of prime ideals, miscellaneous topics in algebraic numbers, cyclotomy, fields of functions, (DICKSON); Hensel's p-adic numbers (WAHLIN). Remarkable uniformity of treatment has been maintained throughout, the style following closely that of Dickson's *History of the Theory of Numbers*. In practically all instances the chief results of the memoir abstracted are concisely stated; there are also occasional indications of the methods followed or short summaries of the proofs. A particularly valuable feature is the correction of such errors in the literature as have been noted by the authors, each of whom is an expert in the topics which he discusses.

Although it would be absurd to attempt a précis of this work which is itself a concise summary and which carries condensation to the limit, it may not be out of place here to note one or two points of general interest, particularly for the encouragement of beginners in this extensive and difficult province of modern arithmetic. First, the relatively large amount of space devoted to cyclotomy is proof that this venerable subject—it was born with Gauss' discovery of 1796—is still lusty. Indeed several of the reports hint at unexhausted lodes where a novice might yet dig out nuggets of value. Quite recently, for example, Burnside has considered octosection. These smaller finds are valuable chiefly as whetters of the appetite for discovery; the mother lode lies elsewhere. Again, the fascinating highway of Hensel's p-adic numbers, but lately opened up to general traffic and bearing as it does toward analysis, offers an attractive and less hazardous entrance to the entire theory than do the older, strictly arithmetic approaches. The literature of this subject has a refreshing newness; it has not yet grown dishearteningly vast, nor is it overloaded with minute and harrassing technicalities. To reach quickly the heart of the classical theory by well travelled roads one cannot do better than follow Landau's exposition (Teubner, 1918), as yet unsurpassed for brevity and clearness.

Although, as can be seen from an inspection of this Report and of current literature, the theory of algebraic numbers owes most of its extraordinarily rapid development to the German school, evidence is not lacking that at last the mathematical public in America, England, and France is becoming aware of the splendid creations of Kummer, Dedekind, Kronecker, Hensel and their followers. The contributions of American mathematicians have been few but choice. To cite only three, all mentioned in the Report, we may recall the fundamental and far reaching researches of E. H. Moore and Dickson on the classes of residues of a prime ideal modulus, Mitchell's correction of errors in the work of Kummer which escaped the notice of so penetrating a mind

as H. J. S. Smith, and Vandiver's numerous contributions to the same end. Last there is the recent work of Dickson, which bids fair to be epoch making, on *Algebras and their Arithmetics*,* where the classic theory of algebraic numbers finds a simple and profound generalization.

With the books of Landau and Dickson, the report of Hilbert and that of the present authors now available, it is to be hoped that algebraic numbers, one of the major divisions of modern mathematics, will not much longer remain in learned obscurity, but will take its rightful place as one of the chief glories of any liberal mathematical education.

E. T. Bell

VOLUMES II AND III OF DICKSON'S HISTORY

History of the Theory of Numbers. By L. E. Dickson. Vol. II, *Diophantine Analysis*, xxv + 803 pp. Vol. III, *Quadratic and Higher Forms*, (with a chapter on the class number by G. H. Cresse). iv + 313 pp. The Carnegie Institution, Washington, 1923.

Since the time of Gauss, the theory of numbers has developed in a number of different directions. Let us examine this development prior to the year 1890. Dirichlet and Riemann founded the analytic prime number theory; Kummer, Kronecker, and Dedekind created the theory of algebraic numbers; Eisenstein, Hermite, Smith, and Minkowski developed the arithmetic theory of forms; Jacobi, Eisenstein, Kronecker, Smith, and Hermite applied the theory of elliptic functions to various problems. It will be noted that, in the main, this progress centered about a few great names. The discoveries of these men did not excite the attention of other mathematicians in many cases because the contents of the original papers were often complicated and difficult to read, and few suitable texts were provided to meet the needs of the beginner.

In considering the period between 1890 and 1900, however, a decided change is noted. In this interval appeared the *Lehrbuch der Algebra* of Weber and Hilbert's *Bericht über die Theorie der Algebraischen Zahlen*. These works and the original papers of the same authors appear to have exercised a profound influence on a number of able young mathematicians. In another line, Hadamand and de la Vallée Poussin obtained epoch making results in the theory of the Riemann zeta function, with applications to the asymptotic distribution of prime numbers. Minkowski founded a geometry of numbers which has bearing on many parts of the number theory. Dickson initiated his extensive contributions to the subject by developing the theory of finite fields,

* This sentence was written by the reviewer before the award to Professor Dickson of the Cincinnati Prize. See page 90 of this issue. The Editors.

having direct connection with the theory of congruences, and attacked arithmetic problems arising in the theory of linear groups.

Since 1900 the theory of numbers has been investigated not only by a large proportion of mathematicians of the first rank, but by a great number of less gifted individuals, until at the present time it would perhaps be unwise to assert that the subject has not received sufficient attention from men of research.

The progress has been continuous, and, it seems to the writer, comparable with that during any similar interval of time in the past. In particular, it does not appear unlikely that the year 1909 will be referred to in the future as a landmark in the theory of numbers. In that year A. Thüe published his proof of the theorem that the equation

$$f(x, y) = c,$$

where $f(x, y)$ is a homogeneous binary form of degree n with integral coefficients, irreducible in the rational field, and c is an integer, has only a finite number of solutions if $n > 2$; Hilbert's proof of Waring's theorem that any integer can be expressed as the sum of not more than k nth powers, k being an integer depending on n only, appeared; Wieferich showed that if

$$x^p + y^p + z^p = 0$$

has a solution in integers, x, y, and z prime to the odd prime p, then

$$2^{p-1} \equiv 1 \pmod{p^2};$$

and Furtwängler announced and proved a general theorem of reciprocity between two integers in a relative cyclotomic field of which the law of quadratic reciprocity between two odd rational integers is a special case. Each of these results attracted much attention and was the starting point of investigations of wide scope.

In recent years the subject has developed rapidly and in various ways. Based on important discoveries by Hecke, Landau and others regarding the Dedekind function, the methods employed in treating the theory of the distribution of rational primes have been extended in such a way as to yield corresponding theorems concerning the distribution of ideal primes in an algebraic field. Furtwängler and Fueter developed the theory of the class field (Klassenkörper) of an algebraic field, which has close connection with the higher laws of reciprocity and the complex multiplication of elliptic functions. Dickson has been the leader in applications of modern algebraic methods to various problems; has founded theories of modular invariants, and of arithmetics of linear associative algebras, the latter having important uses in ordinary Diophantine analysis. Hardy and Littlewood have discovered new analytic methods in additive number theory, obtaining therefrom remarkable results concerning partitions, representations of a number as sums of

squares, Waring's theorem (referred to above) and the representation of an integer as the sum of primes. Humbert, Mordell, and others have contributed extensively in recent years to the arithmetic theory of forms.

It is fitting, during such an interesting stage in the development of number theory, that the monumental work under review should appear. Volume I of the *History* was reviewed by D. N. Lehmer in this BULLETIN (vol. 26 (1919), p. 125).

The plan of the history is to cover all parts of the subject with the exception of algebraic numbers. Volume IV, which has not as yet appeared, is to comprehend, in particular, the extensive literature on the laws of quadratic reciprocity.

For the second volume Dickson has given a resumé of the contents of each chapter in the preface; in the third volume a resumé is at the beginning of each chapter. In view of this we shall not take up in detail the material covered, aside from a few points which expecially interest the reviewer.

The chapter headings of volume II are: Polygonal, pyramidal and figurate numbers; Linear Diophantine equations and congruences; Partitions; Rational right triangles; Triangles, Quadrilaterals and tetrahedra; Sum of two squares; Sum of three squares; Sum of four squares; Sum of n squares; Number of solutions of quadratic congruences in n unknowns; Liouville's series of eighteen articles; Pell equation; Further single equations of the second degree; Squares in arithmetic or geometric progression; Two or more linear functions made squares; Two quadratic functions of one or two unknowns made squares; Systems of two equations of degree two; Three or more quadratic functions of one or two unknowns made squares; Systems of three or more equations of degree two in three or more unknowns, Quadratic form made an nth power; Equations of degree three; Equations of degree four; Equations of degree n; Sets of integers with equal sums of like powers; Waring's problem and related results; Fermat's last theorem,

$$ax^r + by^s = cz^t,$$

and the congruence

$$x^n + y^n \equiv z^n \pmod{p}.$$

Volume III contains: Reduction and equivalence of binary quadratic forms; Representation of integers; Explicit values of x, y in

$$x^2 + \Delta y^2 = g;$$

Composition of binary quadratic forms; Orders and genera and their composition; Irregular determinants; Number of classes of binary quadratic forms with integral coefficients; Binary quadratic forms whose coefficients are complex integers or integers of a field; Number of classes of binary quadratic forms with complex integral coefficients; Ternary quadratic forms; Quaternary quadratic forms; Quadratic forms

in n variables; Binary cubic forms; Cubic forms in 3 or more variables; Forms of degree $n \geq 4$; Binary Hermitian forms; Hermitian forms in n variables and their conjugates; Bilinear forms, matrices, linear substitutions; Representation by polynomials modulo p; Congruencial theory of forms.

On reading the chapter entitled "Pell's Equation," which cites 314 references, one is impressed by the number of writers who independently obtained special methods of solution approximating in character the well known continued fraction algorithm. This seems particularly noteworthy as an example of the evolution of a mathematical idea.

Those chapters in volume II beginning with "Squares in arithmetic or geometric progression" and ending with "Equations of degree n," seem to the writer to be a very unusual contribution to mathematical literature. They deal with subjects which particularly interest amateur investigators and a large proportion of the references are to be found in the problem columns of lesser known publications. Nothing has ever appeared which compares to these chapters in furnishing information about special Diophantine problems.

It will be seen from the above list that the material treated in the third volume is much more technical, in the main, than that taken up in the other volumes and the form in which some of the chapters have been written indicates that more space has been devoted to explanations of various general theories. The chapter on the binary quadratic form class number contains 375 references, indicating the fascination which this topic has had for numerous writers. Legendre, Gauss, Jacobi, Cauchy, Dirichlet, Eisenstein, Smith, Dedekind, Liouville, Hermite, Kronecker, Poincaré, Weber and Klein are some of the well known names associated with it.

Perhaps less known than the material in the other chapters is that contained in the chapters on cubic and Hermitian forms. Following Eisenstein, consider the cubic form

$$f = (a, b, c, d) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

with rational integral coefficients, and set

$$F = Ax^2 + Bxy + Cy^2, \; A = b^2 - ac, \; B = bc - ad, \; C = c^2 - bd.$$

The *determinant* of the quadratic form for $2F$ is called the determinant D of the cubic form f.

Using linear transformation, the terms proper, improper, equivalent, and class are defined for f in a manner analogous to that in quadratic form theory. Also, correspondences are obtained between the classes of F and certain classes of f. The representation of an integer by f depends on finding integers representable by F, (if $(A, B, C) = 1$),

whose cubes can be represented in the form

$$V^2 - DV^2.$$

(Cf. the *History*, Chap. XII, Eisenstein[2], Pepin[13]). The topic has been investigated extensively by Eisenstein, Arndt, and Pepin.

After Hermite we write

$$v = x + iy,\ v_0 = x - iy,$$

and u_0 for the conjugate imaginary of u, and we have the binary Hermitian form

$$f(v, u;\ v_0, u_0) = Avv_0 + Bvu_0 + B_0 v_0 u + Cuu_0$$

where A and C are real, while B, B_0 are conjugate imaginaries. Then f takes only real values under a linear transformation and $BB_0 - AC$ is an invariant, called the determinant of f. Definite, indefinite, and reduced forms are defined and a theory has been built up which is analogous to that of binary quadratic forms. The subject has applications to the theory of quaternary quadratic forms and has been developed mainly by Hermite, Picard, Bianchi and Humbert.

We shall now consider the place of the work in mathematical literature. On page xx of the preface of volume II the author says:

"Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history." In the reviewer's opinion, Dickson has come very close to realizing these ideals, and this circumstance gives the work its greatest value. In particular, it appears that he has had always in mind the needs of the investigator. An endeavor is made to supply every scrap of information that might possibly aid in research on a particular problem.

It is characteristic of number theory that from time to time results which are quite new and of an entirely elementary character are obtained by methods also elementary. This being the case, one often finds it of value to study papers as old as those of Euler, Lagrange and Legendre, as *all* of their ideas have not yet been incorporated in general theories, but are, however, reported on in full in the *History*.

Muir's *History of the Theory of Determinants*, which resembles Dickson's *History* closely in giving detailed reports of each paper cited, differs, however, in other respects. Much space is devoted by Muir to

criticising the results of articles or in praise of various writers. This type of material does not, in general, particularly interest the man of research, and the published opinions of individuals as to the value of mathematical results rarely have any interest after a term of years. This brings out a quality of Dickson's *Hystory* which it seems well to emphasize, namely, that these volumes may be regarded as a permanent foundation on which to bring the history of number theory up to date at various intervals in the future. It is hardly conceivable that anyone should want to re-examine all the material that Dickson has already gone over and re-write his work. He has put things in such form that no more is perhaps needed in order to cover future progress in number theory than to make additions to his work from time to time.

The material shows in itself how useful the history will be in enabling investigators to avoid duplication in published results. It is to be noted in glancing over the pages, how often the contents of a published article, is, unknown to the writer of it, largely a repetition of previous research of some other author.

After some experience in the use of the first and second volumes, the reviewer has noted a number of instances, where the author has expressed in a nut-shell the main results of a long and involved paper in a much clearer way than the writer of the article did himself. The ability to reduce complicated mathematical arguments to simple and elementary terms is highly developed in Dickson.

It often happens in the history of mathematics that a mathematician becomes a specialist in a particular topic, and, after years of experience with it, he publishes a treatise giving a harmonious and comprehensive development of the subject, the material being all arranged and presented according to his own particular point of view. This treatise may become a classic and its readers are likely to get in the habit of ignoring, to a considerable extent, the literature that preceded its publication. In this way the points of view of the older writers are often lost sight of, as these treatises rarely, if ever, reproduce all the older material on a particular topic. It would seem that there is too great a preponderance of books of this sort in the literature and too few histories or reports of the type of Dickson's work.

It may happen that some readers of this review are not interested particularly in number theory, but are specialists in some other field. Let them consider what a work of similar character to Dickson's *History* on their favorite subject would mean to them. Perhaps they will then be convinced of the supreme importance of producing detailed histories or reports in all branches of mathematics.

H. S. VANDIVER

THREE BOOKS ON RELATIVITY

The Mathematical Theory of Relativity. By A. S. Eddington. Cambridge, University Press, 1923. VI + 247 pp.

The Meaning of Relativity. By Albert Einstein. Translated into English by E. P. Adams. Princeton University Press, 1923. 123 pp.

The Theory of General Relativity and Gravitation. By L. Silberstein. New York, Van Nostrand, 1922. IV + 141 pp.

In attempting a combined review of these three books, I do not wish to give the impression that each does not merit extensive consideration. However, there are necessarily many points of similarity, and by pointing these out and emphasizing the differences I may be able to give an idea of the character of each book. Eddington's book is by far the most comprehensive and contains practically all of the mathematical treatment appearing in the other books, but the latter contain many helpful and stimulating observations and interpretations.

Einstein recalls in his first lecture some of the more fundamental ideas and equations of pre-relativity physics, and converts them into tensor form. In the second lecture the equations for the same physical concepts as interpreted in special relativity are given tensor form. This should be particularly helpful to those, who, not being entirely familiar with mathematical processes, have tried to acquire a knowledge of tensor analysis from the general treatments, such as was given by Einstein in his 1916 paper, and is largely followed in the books under discussion (Eddington, Chapter 2; Einstein, Lecture 3; Silberstein, Chapter 3). After following the first two lectures the reader begins to feel that, as Eddington says (3), "our knowledge of conditions in the external world, as it comes to us through observation and experiment, is precisely of the kind which can be expressed by a tensor and not otherwise." He may not yet be prepared to agree with the last part of this statement, but as he comes to appreciate the effective use which Einstein has made of tensor calculus in his general theory of relativity, he is forced to the conclusion that here is a great contribution to mathematical physics. Scientists may agree or not with Einstein's interpretation of his equations as regards the character of physical space and, in particular, the significance of the well known crucial tests of his theory, but they cannot afford to ignore the guidance of tensor calculus in their attempts to give mathematical formulation to the results of experiment.

The postulates and ideas of special relativity are set forth, more or less briefly, by all three authors in preparation for the transition to general relativity. In their generalized form the postulates may be stated in the explicit form:

1. Everything connected with location which enters into observational knowledge—everything we can know about the configuration of events—is contained in a relation of extension between pairs of events; this relation is called the interval and its measure is denoted by ds; in any system of coordinates $ds^2 = \Sigma g_{ij}\, dx^i\, dx^j$, where g_{ij} are functions of the coordinates which describe the metrical relations of the space-time continuum and also the gravitational field.

2. The path of a freely moving particle, whether in the presence or absence of a gravitational field, but not in an electromagnetic field, is a geodesic of the quadratic form ds^2.

3. The track of a light-wave satisfies the condition $ds = 0$. (Ed. 10, 36; S. 20; E. 71, 87, 102.)

The generalization consists in replacing inertial, or Galilean, systems by general systems of coordinates and in interpreting the g's as potentials of the gravitational field. As a first step in this and other generalizations, we have Einstein's postulate:

4. For infinitely small regions there will be an inertial system (i. e. a local reference system for which the g's have the Galilean values) relative to which the laws of special relativity are valid. (E. 70; S. 12.) However, the assumption underlying the generalization is Einstein's "principle of equivalence". Eddington (40) considers the latter to be "a hypothesis to be tested by experiment as opportunity offers"; ... "as a suggestion rather than a dogma admitting of no exceptions." Under the guidance of this principle Einstein was led to the postulates:

5. Inertial mass and gravitational mass are identical.

6. The law of gravitation for empty space is $R_{ij} = 0$, or $R_{ij} = \lambda g_{ij}$, where λ is a very small constant. It may be that this is equivalent to the "principle of equivalence," in other words, that the principle breaks down for all other curved worlds.

Proceeding from these postulates, Eddington (83-90) and Silberstein (92-100) derive the Schwarzschild solution of the equations $R_{ij} = 0$ for a radially symmetric field and obtain the equations of the geodesics which are identified with the paths of the planets about the sun; from these are derived Einstein's formula for the motion of the perihelion of Mercury; Einstein (105-07) merely states these results. Eddington adds to Postulate 3 the requirement that the tracks of light be geodesics; and, by allowing ds to approach zero in the preceding discussion, he obtains the formula for the deflection of light; Silberstein gives a similar derivation and also states that it is a consequence of Fermat's principle, as shown by Levi-Civita and de Sitter. Einstein does not assume that the paths of light are geodesics. He adheres to Postulate 3 and derives (103) the formula for deviation of light from an approximate solution of the equations

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -kT_{ij}. \tag{1}$$

From this solution he determines the value of the constant k and upon it he bases also his conclusion concerning the shift of spectral lines; the discussion of this question by Eddington and Silberstein is based on the expression $ds^2 = (1 - 2m/r)\,dt^2$, obtained from the Schwarzschild form for an atom at rest or practically so.

Before giving the results for planetary motion as stated above, Einstein (88) considers an approximate solution somewhat after the manner of his 1916 paper. Silberstein (35–8) follows a similar course and obtains the equations given by Einstein with added terms. He thinks that Einstein failed to obtain these "through a too-hasty computation of the Christoffel symbols." It seems to be a question rather of what is meant by "first approximation." As interpreted by Einstein in both places these added terms vanish. In any case the interpretation of these terms as due to an acceleration exerted upon the frame of reference by a velocity field cg_{ii} $(i = 1, \ldots, 4)$ is interesting.

Einstein's derivation (90–94) of equations (1), as a generalization of the Laplace-Poisson equation,

$$\Delta\varphi = 4k\pi\rho, \tag{2}$$

is one of the most interesting parts of the book. The tensor T_{ij}, the energy-momentum tensor of matter, "includes in itself the energy density of the electromagnetic field and ponderable matter", ... "It is only the circumstance that we have not sufficient knowledge of the electromagnetic field of concentrated charges that compels us, provisionally, to leave undetermined in presenting the theory the true form of this tensor." However, in accordance with special relativity, the principle of conservation of momentum and energy is expressed by the vanishing of the divergence, that is

$$\frac{\partial T_{ij}}{\partial x^i} = 0. \tag{3}$$

This and Postulate 4 serve as a guide in the determination of the left-hand member of (1). Einstein does not say that this is equivalent to

$$T^{ij}_{\ \ i} = 0 \tag{4}$$

in general coordinates, where the subscript indicates covariant differentiation, but he explicitly states (91) "we shall have to assume (4) as valid." We wish to emphasize that there is an assumption involved in replacing ordinary derivatives of special relativity by covariant derivatives in general relativity—it is the assumption that in applying Postulate 4 the coordinates at the point are geodesic. Eddington (119) makes the transition from (3) to (4) by an appeal to the "principle of equivalence." Since the components g_{ij} are interpreted as the potentials of the gravitational field—a fundamental assumption of the Einstein theory—the generalization of equation (2) requires of the left-hand member of (1) that it contain no differential coefficients of g_{ij}

higher than the second, that it be linear and homogeneous in the second differential coefficient, and that its divergence be zero. Einstein states without proof that the only tensor of the second order satisfying the first two conditions is $R_{ij} + ag_{ij}R$, in the usual notation, where a is an arbitrary constant.* He then shows that the third condition requires that $a = -1/2$, and in doing so makes use of a particular type of geodesic coordinates at a point. In view of his assumption mentioned above, he is then justified in his statement that the vanishing of the divergence is proved for any system of coordinates. Eddington (119) and Silberstein (81) establish the same result by making use of the four fundamental identities

$$R^i_{ij} = \tfrac{1}{2}\frac{\partial R}{\partial x^j}.$$

So far as I know, these identities were first established by Levi-Civita (RENDICONTI LINCEI, 1917, p. 388) in a general way by means of the identities of Bianchi; Eddington's proof (115) is based upon the use of geodesic coordinates.

The tensor T_{ij} for an electromagnetic field alone was written by Einstein in his 1916 paper (§ 20) in the form

$$T^i_j = -F_{j\alpha}F^{i\alpha} + \tfrac{1}{4}\delta^i_j F_{\alpha\beta}F^{\alpha\beta}, \tag{5}$$

where $F_{j\alpha}$ is the curl $\partial\varphi_j/\partial x^\alpha - \partial\varphi_\alpha/\partial x^j$ of the electromagnetic potential, δ^i_j are the Kronecker deltas, and

$$F^{\alpha\beta} = g^{\alpha i}\, g^{\beta j}\, F_{ij},$$

using the conventional notation. In his second lecture (53) he shows that this expresses the principles of energy and momentum as developed by Maxwell, and that the four-dimensional formulation of special relativity serves as the guide to the amalgamation into a tensor. By similar considerations Maxwell's equations are written in the tensor form

$$\begin{cases} \dfrac{\partial F_{ij}}{\partial x^k} + \dfrac{\partial F_{jk}}{\partial x^i} + \dfrac{\partial F_{ki}}{\partial x^j} = 0, \\ \dfrac{\partial}{\partial x^j}(\sqrt{-g}\,F^{ij}) = \sqrt{-g}\,\rho_0\dfrac{dx^i}{ds}, \quad \text{or} \quad (F^{ij})_j = \rho_0\dfrac{dx^i}{ds}, \end{cases} \tag{6}$$

where ρ_0 is the proper density of electricity and $(F^{ij})_j$ is the covariant derivative. (Cf. Ed. 170–5; S. 106–113.) In this generalization from special relativity by means of Postulate 4, there is the added assumption that the coordinates are geodesic. Applying covariant differentiation to (5), we have (Ed. 182; S. 121) for the divergence of T^i

$$T^i_{ji} = F_{j\alpha}\rho_0\frac{dx^\alpha}{ds}. \tag{7}$$

Thus the divergence vanishes in regions outside of charged particles,

* For a proof the reader is referred to Weyl, *Space, Time and Matter*, p. 315; also Birkhoff, *Relativity and Modern Physics*, pp. 209–221.

"the only regions in which we can believe that we have the complete expression for the energy tensor", according to Einstein (55). The expression in the right-hand member of (7) is the ponderomotive force four-vector which replaces the electric force vector in the equations of electrostatics, when these equations have been put in tensor form. These equations were derived for special relativity by Einstein in his 1905 paper and again in his second lecture (52). They agree to a high degree of accuracy with the results of experiments on β-particles; and, when written in the form for general relativity, they show that the paths of an electron are not geodesics. Eddington (189) "wants to know what the electron is trying to accomplish by deviating from a geodesic". He considers the field outside the electron and assumes that the field within the electron counterbalances it; he then concludes that an electron in an external field of force having the acceleration given by the equations referred to is "a miracle", whatever that may mean.

As previously remarked, Einstein obtained an approximate solution of equations (1) and therefrom reached the conclusions concerning physical phenomena which have become generally known. These solutions were based on the assumption that the potentials g_{ij} have the Galilean values at infinity and differ little from these values in the neighborhood of matter. In 1917 he presented his views on the so-called *cosmological problem*. Mach held that the inertia of any particle depends upon the whole matter in the universe. If this is true, then (1) "the inertia of a body must increase when ponderable masses are piled up in its neighborhood; (2) a body must experience an accelerating force when neighboring masses are accelerated, and, in fact, the force must be in the same direction as the acceleration; (3) a rotating hollow body must generate inside itself a 'Coriolis field', which deflects moving bodies in the sense of a rotation, and a radial centrifugal field as well" (E. 110). Einstein obtained an approximate solution, involving each of these effects, but of magnitude too small to be tested by experiment; in doing so he was led to make the hypothesis that the physical universe, as distinguished from the space-time continuum, is spherical and closed, and he proposed suitable expressions for the functions g_{ij} and for the tensor T_{ij} which were in keeping with this hypothesis. We cannot here discuss adequately the merits and demerits of this proposal, and of the one suggested by de Sitter; the reader will find this done fully by Eddington (155–168) and by Silberstein (124–137).

Suppose that we had not started with the six postulates previously set down, but had merely taken the first one, which may be interpreted roughly as saying that the physical world (space-time) is a Riemannian manifold of four dimensions with a fundamental tensor g_{ij}. From this tensor others may be derived, such as R_{ij}, B_{ijk}, and so on. Accepting the fundamental principle of relativity that physical laws are expressible

by tensor equations, the next step is to find these equations by a principle of identification with the results of experiment. If, as Eddington (119) says, we take "the view that energy, stress and momentum belong to the world and not to some extraneous substance in the world, we must identify the energy-tensor with some fundamental tensor of the world". Since the divergence of $R_{ij} - \frac{1}{2} g_{ij} R$ vanishes, it is natural to try the equation (1) and see what happens. The left-hand member vanishes only when $R_{ij} = 0$; this, then, is the condition for empty space. From (1) it can be shown that the path of a particle in empty space is a geodesic (Ed. 127). By identifying geodesic coordinates at a point with the Galilean coordinates of special relativity, Eddington (178) shows that the path of a light-pulse is a geodesic; it is a question, however, whether this proof is not equivalent to the assumption involved in the identification. We shall not consider the further consequences of this point of view, but will turn our attention to the more general world-geometry, as developed by Weyl and Eddington.

Weyl and Eddington consider a general four-dimensional continuum in the sense of Analysis Situs and define an affine connection for the manifold by means of forty functions Γ^i_{jk}, symmetric in j and k. By considering the change in a general vector A^i as it undergoes a parallel displacement round a small circuit, they are led to a tensor of the fourth order $^*B^i_{jkl}$ which is a generalization of the Riemann tensor. When this tensor is contracted for i and l, the resulting tensor is the sum of a symmetric tensor R_{jk} and the curl of a vector φ_i. Eddington puts $R_{jk} = \lambda g_{jk}$, where λ is a universal constant and takes

$$ds^2 = g_{ij} dx^i dx^j$$

as the metric of the space; he leaves the functions Γ^i_{jk} perfectly general, which is equivalent to having an arbitrary tensor K^i_{jk}, symmetric in j and k. Weyl, on the other hand, specializes his geometry by expressing the Γ's as certain functions of the components of a symmetric tensor g_{ij}, their first derivatives, and the components of a vector, which is equivalent to φ_i mentioned above.*

Before we proceed further with the development of this geometry

* Eddington's world-geometry is essentially the same as the Geometry of Paths as developed by Veblen and myself in a number of articles in volumes eight and nine of the PROCEEDINGS OF THE NATIONAL ACADEMY, and by me in the ANNALS OF MATHEMATICS, vol. 24 (1923). This geometry is not equivalent to the more restricted type considered by Weyl, as Eddington states (243). Furthermore, the equations of the paths are fundamental, which has a bearing on Eddington's remark (216): "It may be asked whether there is any other way of obtaining tensors. besides the consideration of parallel displacement round a closed circuit. I think not ..."

it is advisable to consider its relation to the actual space of physical phenomena. Eddington (197) says "Two possible ways of generalizing our geometric outlook are open. It may be that the Riemannian geometry assigned to actual space is not exact; and that the true geometry is of a broader kind leaving room for the electromagnetic potential vector to play a fundamental part and so receive geometric recognition as one of the determining characters of actual space." ... "The alternative is to give all our variables, including the electromagnetic potential, a suitable graphical representation in some new conceptual space—not actual space." ... "We have then to distinguish between Natural Geometry, which is the single *true* geometry in the sense understood by the physicist, and World Geometry, which is the pure geometry applicable to a conceptual graphical representation of all the quantities concerned in physics." The latter is Eddington's idea of the significance of his geometry and of Weyl's; Weyl at first held that his geometry was the geometry of actual space, but according to Eddington (198, 208) he now holds to the latter view also. Commenting upon the theory of electromagnetism leading to equation (7), Einstein (108) says "This inclusion of the theory of electricity in the scheme of the general theory of relativity has been considered arbitrary and unsatisfactory by many theoreticians. Nor can we in this way conceive of the equilibrium of the electricity which constitutes the elementary electrically charged particles. A theory in which the gravitational field and the electromagnetic field enter as an essential unity would be much preferable." He reiterates this statement in his latest paper (PREUSSISCHE AKADEMIE DER WISSENSCHAFTEN 1923, pp. 32–38) in which he undertakes to present such a theory. However, he gives no indication that, in adopting Eddington's geometry for the basis of his work, he feels that the geometry is merely a graphical representation of the physical world. On the contrary the idea that he is dealing with the actual world seems to be fundamental in all of his writings concerning the theory of relativity.

Weyl, Eddington, and Einstein identify the vector φ_i of their geometries with the electromagnetic potential. Eddington (208) says it is "the electromagnetic potential because that is the way in which we choose to represent the potential graphically." He also identifies $R_{ij}\,dx^i\,dx^j = 0$ as the equation of light-pulses, but he does not prove that this is a consequence of the former identification. In other words, it seems to me that if Eddington's view of graphical representation is adopted and one identification is made, the other must be established. It seems reasonable that in some way φ_i is connected with the electromagnetic potential, but if one is dealing with actual space is it in fact the electromagnetic potential itself, or if one is making a graphical representation, is that the best interpretation to place upon φ_i? Recently I gave (PROCEEDINGS OF THE NATIONAL ACADEMY, June, 1923) another

identification of φ_i based upon the equations of a moving electron. In the present state of the theory, a statement that this is the correct interpretation would be dogmatic. I mention it merely for the purpose of pointing out that if it is found that the conclusions of Weyl, Eddington, and Einstein are not in accordance with physical phenomena, as they become more clearly understood, the difficulty may be due to their identification of the vector φ_i.

It would not be proper to bring this review to a close without saying something about the Hamiltonian principle of action. Lorentz and Hilbert seem to have been the first to try to take over this principle from classical mechanics and determine its bearing upon general relativity. Einstein considers it in his 1916 paper (§ 15) and later in the same year made a more extensive study of it. Silberstein (88) feels that "the importance of the Hamiltonian principle seems to be unduly overestimated," and Eddington, who gives (131–144) a full discussion of it, together with what he calls the Hamilton derivative, says (138) "We have thus to remember that when a writer begins to talk about action, he is probably going to consider impossible conditions of the world. (That does not mean that he is talking nonsense—he brings out the important features of the possible conditions by comparing them with the impossible conditions)." There are two distinct methods of approaching this question. One is to build up an integral such that on equating its variation to zero, known results are obtained. This is what Weyl has done (*Space, Time, and Matter*, pp. 209–216; pp. 230–237). The other is to start with the idea that an integral can be found which will be the philosopher's stone to reveal the hidden treasures of the physical world. Thanks to the guiding hand of the tensor calculus the search for such an integral (if it exists) is not a game of blind man's buff. Weyl (295) chooses an integral and discusses the consequences of the choice, although he questions whether his action-principle is realized in nature exactly in the form chosen. Eddington (232) suggests and considers several integrals, one of which Einstein develops in his recent paper in such a manner as to obtain an expression for the Γ's in terms of a tensor s_{ij} and a vector i_j. The results of these theories await verification. It may be that one of them is the hoped-for *omnium gatherum* with or without the correct identification.

L. P. EISENHART

WINGER ON PROJECTIVE GEOMETRY

An Introduction to Projective Geometry. By R. M. Winger. Boston, D. C. Heath, 1923. V + 443 pp.

Among the books on projective geometry in English which have appeared so far, Winger's *Introduction* is a distinct novelty. In fact it justly breaks away from the more or less traditional Cremona-Reye style of a passed period and presents, on the whole, such topics which for a more advanced study of geometry are of essential importance. For this reason, the reviewer is glad to declare from the start that Winger has written an excellent text-book.

The author himself says that "this book is intended as an introductory account for senior-college and beginning graduate students — for the prospective teacher who is seeking proper orientation of elementary mathematics, as well as the university student who lacks the preparation for an intelligent reading of the general treatises on higher geometry and the modern books on higher algebra". As mathematical preparation for a proper understanding of the book collegiate training in algebra, analytic geometry and calculus is all that is required. It is perhaps not necessary to review in detail the thirteen chapters which in order deal with essential constants; duality; the line at infinity; projective properties; double ratio; projective coordinates; the conic; collineations and involutions in one dimension; binary forms; algebraic invariants; analytic treatment of the conic; collineations in the plane; cubic involutions and the rational cubic curve; non-euclidean geometry. The book is very clearly and expressively written throughout, and the propositions are stated concisely and in simple straightforward English. Undoubtedly it will have a very refreshing effect upon the student.

The few criticisms which the reviewer wishes to make concern in some instances the method of presentation rather than the choice of contents. Thus, on page 8, the statement "to show that a condition is necessary and sufficient entails the proof of a proposition and its converse" is, of course, not obvious and needs qualification. In the definition of isotropic lines by $x \pm iy - k = 0$, it would probably be better not to introduce the new term *circular rays* for the special case when $k = 0$. The old term *ray* (*Strahl*) which has an optical meaning when qualified should be abandoned. The word *line* (straight) is sufficient for this purpose. Thus one might speak of $x \pm iy = 0$ as the principal isotropic lines. The paradoxical statements at the top of page 54 might have been omitted without harm; since from a purely geometrical standpoint they are absolutely meaningless.

In Chapter IV on projective properties the statement that "between a figure and its projection there exists the relation of *one-to-one-correspondence*" should be qualified as *algebraic*, in order to be projective, since it is not difficult to establish non-projective one-to-one-correspondences, as for example between the x- and y-axis of a cartesian system by means of $y = \sqrt[3]{x}$ when restricted to real numbers.

Discussing metric and projective properties the author says that "since a circular cone is merely a perspection of a circle from the vertex it follows that any section, not on the vertex; i. e., any proper conic is a projection of the circle". Now there is certainly no obvious conclusion impelled by such an assumption. No matter whether the spaces in which one operates are metrically or projectively defined, such a statement needs proof.

The treatment of projective coordinates is in the opinion of the reviewer the least satisfactory in the whole treatise. The term projective applied to any mathematical doctrine means independence of any particular metrical concept whatsoever. Hence from a scientific standpoint, the establishment of projective coordinates should not be based either on euclidean or cartesian space, even when disguised in the form of a homogeneously made cartesian space. The proper starting point is the famous, indefinitely repeated harmonic quadrangle construction of v. Staudt, plus Dedekind's axiom of continuity, etc., and the establishment of a linear (1,1)-correspondence between the elements thus obtained and the continuum of real numbers. Thus on a line every point A_i is individualised by a parameter $\lambda_i = x_1/x_2$. Projectively, the distance between two points A_1 and A_2 is defined by $A_1 A_2 = \lambda_2 - \lambda_1$. A similar procedure leads to the definition of projective coordinates of spaces of two, three, or, in fact, of any number of dimensions. We easily obtain more general spaces by admitting complex values for the parameters λ. In this manner, a perfectly rigorous projective geometry may be built up without the least reference to euclidean or cartesian metrics. Also from a didactic standpoint this procedure is simple enough and does not offer greater difficulties to the student than other more or less metric methods. In a purely projective geometry (of the plane) the statement that "the projective coordinates of a point are proportional to fixed multiples of the distances of the point from the sides of the triangle of reference" has no place at all.

The chapters on binary forms and algebraic invariants are very well done. In courses on the algebraic theory of invariants the greater part of the time is usually spent in the study and assimilation of the purely formal algebraic processes, so that little time is left for geometric applications. Winger's presentation of the subject, based on the polarizing process, may be called ideal from the standpoint of the geometrician. The treatment of apolarity in the ternary field, however, must seem rather

abrupt and disconnected to the student, since it is based merely on a suggested generalization from the binary field.

"Absolute coordinates," defined by

$$x = X + iY, \ \bar{x} = X - iY,$$

might better be called *isotropic coordinates*, as introduced by Laguerre. If one speaks of isotropic lines it would seem natural to speak also of isotropic coordinates. The term *absolute* introduced by Cayley is not expressive of the idea it intends to convey. There certainly is nothing "absolute" about such real or imaginary boundary conics of non-euclidean geometry.

The chapter on collineations brings out the important properties of these transformations. Involutions on the rational cubic curve are treated in a very instructive manner. The idea of order and class of a curve resulting from the principle of duality is illustrated effectively by a number of well chosen examples. The restriction of involutive properties to rational curves reflects, of course, the author's preoccupation with such curves in his researches on self-projective rational curves. On the other hand, the elliptic cubic, as is well known, offers a beautiful example for the application of projective theories.

In the concluding chapter we find a short account of non-euclidean geometry in the plane after the style of Klein, and a historic note thereupon. That a historic account in a book on projective geometry should be limited to non-euclidean geometry seems somewhat peculiar. It would have been perfectly proper to give a general historic sketch on the entire subject of projective geometry.

It is proper to point out an error which is very common in histories of mathematics and which is contained in the following statement on page 420. "Little progress was made until about a hundred years later when Gauss (1777-1855) and his friends and pupils became deeply interested in the subject." Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatschewsky and Bolyai. As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and assumed a rather skeptical attitude towards the new discoveries. Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory.

A very valuable feature of Winger's introduction is the number of well chosen exercises. Altogether the book will serve a very useful purpose. It will enable the student to acquire the basic concepts and theories which are nowadays a necessity for students who wish to enter into the study of more advanced geometric theories. In this sense the book deserves to be very highly recommended.

ARNOLD EMCH

SHORTER NOTICES

The Making of Index Numbers. By Irving Fisher. Boston and New York, Houghton Mifflin Company, 1922. xxxi and 526 pages.

This book is an elaborate inductive study of price index numbers, the prime object being the discovery of the best, or at least the most accurate index number. To begin with, six fundamental types are considered, the arithmetic, geometric, harmonic, median, mode and aggregative. Each of these may be weighted in different ways, thus giving rise to 24 non-identical formulas. According to Fisher an index number should satisfy two chief tests. Since an index number implies two dates, one of which is the base year, the interchange of years yields a second formula called the *time antithesis* of the first. If P_{01} represents the first index number, the time antithesis is $1/P_{10}$, and the first test or *time reversal* test is $P_{01} \cdot P_{10} = 1$.

The second test, original with Professor Fisher, is called the *factor reversal* test, and assumes that index formulas should give consistent results if applied to prices and to quantities. That is

$$P_{01} \cdot Q_{01} = V_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_0}$$

where p_0, p_1 represent prices and q_1, q_0 quantities in the years 0 and 1 respectively. Q_{01} is obtained from P_{01} by interchanging the p's and q's. V_{01} is called the value index. $V_{01} \div Q_{01}$ is called the *factor antithesis* of P_{01}.

Each time antithesis and each factor antithesis is an index number, and after dropping out identicals, 22 new formulas are added to the list making now a total of 46. But only four of these satisfy the first test and none the second. Taking the geometric mean of a pair of time antitheses or of a pair of factor antitheses gives an index number which conforms to the first or to the second test respectively. Performing both operations gives a number which satisfies both tests. By these means the number of formulas is increased to 96 and forms the main series of index numbers discussed in the book. Certain other supplementary formulas giving slight variations to those in the main series are derived by a process called "crossing the weights" and bring the total to 134.

These 134 index numbers now go through a sifting process to find the best one. First all unweighted formulas and all formulas based upon the median or the mode are ruled out because of freakishness

or lack of sensibility. Others are dropped because of more or less bias in one direction, leaving 47 fairly good index numbers. In testing the formulas, the prices and quantities of 36 commodities in the six rather extraordinary years from 1913 to 1918 are used. A surprising thing is the close agreement of this group of index numbers, thus giving the author one of his answers to the argument that the index number to be used should depend upon the purpose for which it is to be employed. Further sifting yields 13 index numbers all satisfying the two tests. Of these the formula

$$\sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \cdot \frac{\sum p_1 q_1}{\sum p_0 q_1}}$$

"is at least equal in accuracy and is probably slightly superior in accuracy to any of the others". This index number is called by Fisher the "ideal" index number, but will probably go into the literature as *Fisher*'s index number, although he insists that the names Walsh and Pigou should be used also.

Eight other numbers are given honorable mention. One in particular, on account of its rapidity of calculation, accuracy and simplicity

$$\frac{\sum (q_0 + q_1) p_1}{\sum (q_0 + q_1) p_0}$$

is really to be preferred to the ideal except when the utmost accuracy is desired, notwithstandig the fact that it is not included in the group of 13 satisfying both tests.

This brief summary gives but a poor idea of the elaborate detail of the book, written as it is for the general reader as well as the specialist. There are worked out examples, tables of comparisons, graphical explanations, notes, appendices for mathematical details until no point is left which is not thoroughly discussed. Chapter 15 on the speed of calculation is a fair example of the painstaking work throughout the book. In this chapter the index numbers for prices and quantities of the above mentioned 36 commodities for the years 1914–1918 were computed, timed and ranked for each of the 134 formulas. The times varied from one hour to 64.5 hours. The ideal formula ranked 29 and took 14.3 hours to calculate. The second of the formulas mentioned above ranked 16 and took 9.6 hours.

We cannot say that Professor Fisher has actually proved that his ideal index number is really the best one, if a best one exists. It seems to be one among several good ones. But the book does prove that some of the index numbers in use are bad ones.

A. R. CRATHORNE

Statistical Method. By Truman L. Kelley. New York, The Macmillan Company, 1923. xi + 390 pp.

There is throughout this book such a close coordination between the theoretical developments and the practical applications as to make it fairly obvious that the mathematical problems were suggested largely by the work of the author on practical problems of statistics. The method of the book is inductive, starting with quantitative data to be described, and developing the appropriate mathematical methods for the suitable description and elucidation of various kinds of data. The book seems remarkable with respect to the large number of topics treated in the given space. This seems to have been made possible in part by using in the earlier parts of the text a good many concepts whose meanings are developed later. In general, this plan involves considerable departure from logical sequence, but the method is made practicable by the use of forward references.

The first five chapters of the book together with Chapter VIII on correlation are suitable reading for the beginning student with but little knowledge of mathematics. In these chapters elementary statistical methods are explained for the benefit of biologists, economists, educators, psychologists and others who use statistical data in their work. Moreover the treatment is so well illustrated by concrete examples as to make an appeal to those who have data to analyze, and the book will tend to promote a higher standard of statistical practice in this country. While the beginner will thus find the book of interest, a large part of the book is planned for the advanced student, and he will find here a wealth of material for his purposes, whether his main interest be in the theory of statistics or in applications to fields such as economics, biology, psychology or education.

The book presents a good many new formulas and adaptations of known formulas to particular purposes. In the preface the author expresses the very commendable view that he can see "no value except at times a slightly greater ease of manipulation, in using a measure whose probable error cannot be calculated if one with a known probable error and serving the same purpose exists." In harmony with this view the book gives a large number of determinations of probable errors. The determinations of these probable errors was surely a very difficult undertaking on the part of the author and he should be complimented on his courage. The author requests the critical analysis by fellow statisticians of his determinations of probable errors, and "such charity in reporting shortcomings as may be due one who has acted upon the policy that as shrewd an estimate as possible of the probable error of a statistical constant is better than no estimate at all." While there is much to commend this attitude in discussing applied

problems, it would seem very important to indicate the fact when there is any considerable doubt as to the validity of the formulas for the probable error.

On page 91, there is a statement which implies that the arithmetic mean and the median are the two most important averages. It would seem safer to limit such a statement to certain purposes. For example, the geometric mean is surely very important for some purposes. On page 154 there seems to be an oversight in the usage of the word "slope". The slope being the tangent of the angle with the positive direction of the x-axis, we note that the slope of the one line would be r and of the other $1/r$ when standard deviations are equal. At the top of page 159, a conclusion seems to be drawn without a clear statement of the hypotheses. It would seem better to say: "It can be easily shown that the deviations x and $y - r(\sigma_2/\sigma_1)\, x$ are uncorrelated. If we assume that these uncorrelated deviations are independent in the probability sense, then the probability of the joint occurrence of the assigned deviations is $z = z'\, z''$. In the chapter on frequency curves, the book follows the ideas of Pearson on generalized frequency curves rather closely. On pages 146–150 the author gives his views of the relation of moments to certain unstable types of distribution.

Nearly half of the book is very properly given to the subject of correlation. The treatment begins with Galton's original ideas, but is extended in many directions and includes the recent developments of the subject. Dr. Kelley's own contributions to the treatment of correlation have naturally given color to the treatment of certain of the advanced aspects of the subject. Special attention should thus be directed to the methods which facilitate the computation of partial and multiple correlation coefficients, including the method of successive approximation devised by Dr. Kelley for finding the regression coefficients when a large number of variables are involved.

The Kelley-Wood table of the normal probability functions with respect to assigned areas is given at the end of the book and will be found useful. The following typographical errors occur:

p. 31, line 13 from bottom, Table I should read Table VII.

p. 31, line 6 from bottom, 65.5 should read 66.5.

p. 33, lines 5 and 6, the words "ordinates" and "abscissas" should be interchanged.

p. 40, chart 16, block labeled 264 should be labeled 246.

p. 77, line 12, .26215 should read .26315.

p. 79, formula 22. The radical sign should extend over the delta square.

p. 79, formula 23a, 5th line from the bottom $2\,\mu^2$ should read μ_2.

p. 89, line 5, $n(n-1)\,(n-2)$ should read $n(n-1)$.

p. 91, line 26, $78.5° - 80°.5$ should read $78°.5 - 83°.5$.

p. 92, line 21, "o" should read "of".

p. 95, formula 46, the last equality sign and the middle minus sign should be deleted and a minus sign should be placed before the last x.
p. 184, line 10, "by formula [12]" should read "by formula [121]".
p. 207, table 39, "test Y reliability .2" should read "test Y reliability .4".
p. 210, formula 161c. The exponents of the r's under the radicals should be deleted.
p. 215, line 19, 5.0—should read 4.0.
p. 240, formula 198, kappa minus 1 in the denominator should read kappa minus 2.
p. 258, formula 213, .6457 should read .6745.
p. 294, table 61, the constants .2200 and .2399 are incorrect.

The reviewer has received most of these corrections from the author.

H. L. RIETZ

Einführung in die mathematische Behandlung der Naturwissenschaften. Tenth edition. By W. Nernst and A. Schoenflies. Berlin, R. Oldenbourg, 1922. xii + 502 pp.

Physics and astronomy are no longer the only pure sciences which depend largely on mathematics. At least thirty years ago the necessity of a knowledge of mathematics for work in other natural sciences had come to be recognized by investigators. Research in theoretical chemistry can no longer be carried on nor understood by one ignorant of the ideas of the calculus, and in other fields the need of a mathematical formulation of problems is increasing.

The book by Professors Nernst and Schoenflies is a text and reference manual for the use of scientists in general and chemists in particular. It represents the mathematical equipment beyond trigonometry which, in the opinion of the authors, should be the possession of the modern chemist. The first edition, which appeared in 1895, was a text on elementary calculus with as much analytic geometry as was necessary for the purpose. In the sixth edition in 1910 there was added some explanation of analytic geometry of space, vectors, foundations of analytic mechanics, and partial differential equations. The tenth edition, appearing about a year ago, contains further material on the theory of heat, relativity, and crystal structure.

From the point of view of the teacher of mathematics the book is written with much less care than are most American texts. But it is altogether probable that a meticulous mathematician could not write a book which would make the same appeal to a student of the natural sciences. The wealth of illustrative material which is possible when it is assumed that the reader has studied some physics and a considerable amount of chemistry gives to the mathematical ideas a reality and vitality which they cannot possibly have otherwise. The examples in analytic geometry are built around the laws of Boyle, Mariotte,

Gay-Lussac, and van der Waals. Although the first problem formulated in symbols of the differential calculus is that of finding the tangent to a parabola, it is preceded by an introduction giving examples of discoveries in natural science which were made possible by calculus, and by a general explanation of the type of problems in science which lead to the idea of rates and the methods of the calculus. The inversion of raw sugar is used as an example to show the meaning of integration before any technique has been developed, and there is an abundance of illustrative material for an indefinite integral. The idea of a discontinuous function is presented by recalling the behavior of the volume of a substance which is being heated. The volume varies continuously up to the melting point, then takes a sudden increase, after which it again varies continuously.

The teacher of college mathematics in America is familiar enough with the complaints of his colleagues in the departments of physics, chemistry, and engineering. They assert, probably with ample justification, that the students cannot use the mathematics which they are supposed to have learned. One remedy for this difficulty might be to give the calculus, perhaps for a second time, at a later stage in the college course when the students had acquired enough background of scientific knowledge to appreciate the meaning of the mathematics. It seems impossible that a student able to understand the illustrative examples in Nernst and Schoenflies should fail to grasp the significance of the mathematical ideas or be unable to use the tools put before him.

At the end of the book is a list of problems for drill work and a collection of formulas beginning with elementary algebra and continuing through the calculus.

Only one error has been noted. On page 117, in equation (8), the constant of integration has not been properly determined. The same equation, however, is given correctly on page 125.

W. R. LONGLEY

Kartenkunde. By Dr. M. Groll. Neubearbeitet von Dr. Otto Graf. Vol. II, *Der Karteninhalt.* Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 133 pp.

The second volume of Groll's Kartenkunde deals with the classification of maps and their contents. Particular attention is given to the various graphic methods of map construction and topographical representation. Copious historic and literary references and illustrations of various methods in various countries add to the interest and value of the second volume. On the whole it deserves the same praise as volume I, which the writer reviewed in this BULLETIN (vol. 29, No. 2 (February 1923), pp. 89–90).

ARNOLD EMCH

Naturwissenschaften, Mathematik und Medizin im klassischen Altertum. By J. L. Heiberg. (Aus Natur und Geisteswelt, No. 370.) Leipzig and Berlin, B. G. Teubner, 1920. 104 pp.

Mathematics and Physical Science in Classical Antiquity. By J. L. Heiberg. Translated from the German by D. C. Macgregor. New York, Oxford University Press, American Branch, 1922. 110 pp.

This little volume from the well known series *Aus Natur und Geisteswelt* (No. 370) maintains the high standards set by the previous volumes of the series. The name of the distinguished author is sufficient guarantee that the contents is authoritative. It is also, however, of peculiar interest, in spite of its brevity and avowedly popular character. The readers of the Bulletin are doubtless all more or less familiar with the main features of ancient Greek mathematics; they are probably acquainted also with the principal results Greek astronomy and the physical speculations of the great philosophers of classical antiquity. It may be doubted, however, if many have ever read a connected account of scientific development as a whole during this period. The reviewer at any rate admits joyfully that much of the contents of this monograph (it is little more than that) was a revelation to him and a very interesting and illuminating one. This must be the justification, if any is needed, for giving space to such an elementary popular little book.

The chapter headings (taken from the English translation) are as follows: I. Ionian Natural Philosophy; II. The Pythagoreans; III. Medicine in the Fifth Century. Hippocrates; IV. Mathematics in the Fifth Century; V. Plato. The Academy; VI. Aristotle. The Lyceum; VII. The Alexandrians; VIII. The Epigoni; IX. The Romans; X. Greek Scientific Literature of the Empire. Byzantium. It will be noted that the arrangement is chronological rather than topical, so that the reader is given for each period a general view of scientific thought at that time. One of the omissions that seems a bit strange is the fact that no mention whatever is made of Lucretius in the chapter on the Romans or anywhere else for that matter.

The English translation is the second volume of the series *Chapters in the History of Science,* which is appearing under the general editorship of Professor Charles Singer. The translation is free rather than literal, and is throughout readable. The reviewer personally dislikes the word "researcher" which appears at frequent intervals, "goings-on" would sound more familiar to American ears than "on-goings" and the "propugnacula" of exact research are a little terrifying, but the original verdict stands: The translation is well done and may be recommended to any one who desires to spend a couple of hours in pleasant reading of interesting material.

J. W. Young

NOTES

The second number of volume 25 of the TRANSACTIONS OF THIS SOCIETY (April, 1923) contains the following papers: *Developments associated with a boundary problem not linear in the parameter,* by R. E. Langer; *Invariant points of a surface transformation of given class,* by J. W. Alexander; *Applications of analysis to the arithmetic of higher forms,* by E. T. Bell; *On the second derivatives of an extremal-integral with an application to a problem with variable end points (supplementary paper),* by A. Dresden; *Abstract group definitions and applications,* by W. E. Edington; *On the integrals of elementary functions,* by J. F. Ritt; *Invariants of the linear group modulo* p^k, by M. M. Feldstein; *On certain polar curves with their application to the location of the roots of the derivatives of a rational function,* by B. Z. Linfield; *Orthogonal systems of hypersurfaces in a general Riemann space,* by L. P. Eisenhart; *Ruled surfaces with generators in one-to-one correspondence,* by E. P. Lane; *Symmetric tensors of the second order whose first covariant derivatives are zero,* by L. P. Eisenhart.

The second and third numbers of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (April and July, 1923) contain: *A class of numbers connected with partitions,* by E. T. Bell; *Note on a new type of summability,* by N. Wiener; *On mediate cardinals,* by D. Wrinch; *Periodic oscillations of three finite masses about the Lagrangian circular solutions,* by H. E. Buchanan; *On certain chains of theorems in reflexive geometry,* by F. D. Sutton; *A poristic system of equations,* by L. B. Robinson; *Systems of two linear integral equations with two parameters and symmetrizable kernels,* by M. Buchanan; *The asymptotic expansion of the functions* $W_{k,m}(z)$ *of Whittaker,* by F. H. Murray; *Some geometric applications of symmetric substitution groups,* by A. Emch; *On elliptic cylinder functions of the second kind,* by S. Dhar.

The second number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *Spherical representation of conjugate systems and asymptotic lines,* by W. C. Graustein; *On a short method of least squares,* by B. H. Camp; *On the convergence of the Sturm-Liouville-series,* by J. L. Walsh; *The functional equation* $g(x^2) = 2ax + [g(x)]^2$, by J. H. M. Wedderburn; *On convergence factors in triple series and the triple Fourier's series,* by B. M. Eversull; *On the minimizing of a class of definite integrals,* by P. R. Rider; *A pythagorean functional equation,* by E. Hille.

At the meeting of the British Association for the Advancement of Science at Liverpool in September, 1923, Sir David Bruce was elected

president for the year 1923—24, to preside at the meeting of the Association in Toronto, in the summer of 1924. The address of the retiring president, Sir Ernest Rutherford, was on *The electrical structure of matter*, and the address of Professor J. C. McLennan, of the University of Toronto, president of Section A, was on the *Origin of spectra*. On the occasion of this meeting honorary degrees were conferred by the University of Liverpool on Sir Ernest Rutherford, Professor Niels Bohr, and Professor J. C. McLennan.

On the occasion of the centenary of Pasteur, the following French mathematicians and mathematical physicists were made officers or chevaliers of the Légion d'Honneur: Andoyer, Brillouin, Cartan, Cotton, Hadamard, Langevin, Vessiot, officers; Baire, Drach, Fréchet, Lebesgue Ollivier, Riquier, Valiron, Villat, chevaliers.

The decision of the first committee of award of the Bôcher Prize of this Society was announced December 28, 1923, at the annual meeting in New York. The award was made to Professor G. D. Birkhoff, of Harvard University, for his memoir entitled *Dynamical systems with two degrees of freedom*, published in the TRANSACTIONS OF THIS SOCIETY, volume 18, pages 199-300.

The prize of $1000 offerred by a member of the American Association for the Advancement of Science for a notable contribution to science reported at the Cincinnati meeting has been awarded to Professor L. E. Dickson, of the University of Chicago. The papers upon which the award was based were presented at sessions of the western meeting of this Society, held in connection with the A. A. A. S.; their titles were: Algebras and their arithmetics, On the theory of numbers and generalized quaternions, and Quadratic fields in which factorization is always unique. The first of these papers, which gives a resume of the work of others and of the authors' own recent work in this field (see p. 65 of this issue), will appear in an early number of this BULLETIN, as will also the third of these papers.

At the annual meeting of this Society in New York, President Veblen announced that the fellowships administered by the National Research Council for physics and chemistry by means of funds from the Rockefeller Foundation had been extended to include mathematics, and that the grant had been increased from $100,000 to $125,000 per annum, effective July 1, 1925. By an action taken since this announcement was made, the inclusion of mathematics as one of the possible subjects will take effect at once, on the fund now operative.

On account of proposed absence from this country, Professor E. B. Van Vleck has resigned as a representative of this Society on the National Research Council. The Council of this Society has selested

Professor H. F. Blichfeldt to complete the term which expires on July 8, 1924, and also to represent this Society for the succeeding period of three years.

At the request of the American Section of the International Mathematical Union, the Council of this Society has selected four additional representatives on the Section, for the nine months including the International Congress, namely, Professors A. B. Coble, Arnold Dresden, L. E. Dickson, and Virgil Snyder.

At the inauguration of Dr. G. J. Trueman as president of Mount Allison University, Sackville, New Brunswick, Canada, the honorary degree of doctor of laws was conferred on Professor R. C. Archibald, of Brown University.

Professor L. A. Howland, head of the department of mathematics at Wesleyan University, has been made acting president of that institution.

Assistant Professor Mary Curtis Graustein, of Wellesley College, has returned to the college after two years' leave of absence.

At St. John's College, Annapolis, Maryland, Professor B. H. Waddell and Assistant Professor T. L. Gladden have resigned, and Mr. G. A. Bingley, instructor in mathematics at the United States Naval Academy, has been appointed assistant professor.

At the Carnegie Institute of Technology, Professor O. T. Geckeler has been made head of the department of mathematics; Assistant Professor H. S. Lightcap has been promoted to an associate professorship, and R. P. Johnson, E. A. Whitman, and E. A. Starr have been promoted to assistant professorships.

At Colgate University, Mr. H. A. Dobell has been granted leave of absence for study at the University of Pennsylvania.

Professor H. P. Kean, of Illinois Wesleyan University, has been appointed assistant professor of mathematics at Wittenberg College, Springfield, Ohio.

At Purdue University, Mr. T. E. Raiford has been granted a year's eave of absence; Dr. W. E. Edington has been promoted to an assistant professorship.

At the University of Georgia, Associate Professor D. F. Barrow has been promoted to a full professorship.

Mr. E. F. Weinberg has been appointed head of the department of mathematics at Rollins College, Winter Park, Florida.

Professor C. H. Richardson, of Georgetown College, Kentucky, has been granted leave of absence for the current year, and is studying at the University of Michigan. Mr. Guy Stevenson is acting for the year as professor and head of the department.

Associate Professor S. Lefschetz, of the University of Kansas, has been promoted to a full professorship of mathematics.

Mr. A. S. Hathaway has been appointed professor of mathematics at Friends University, Wichita, Kansas.

At Iowa State College, Assistant Professor E. C. Keifer has resigned to become head of the department of mathematics at James Millikin University.

Professor D. A. Lehman, of Goshen College, has been appointed professor of mathematics at Bluffton College.

At the Colorado Agricultural College, Mr. C. W. Young has been promoted to an assistant professorship, and Mr. A. G. Clark, instructor in mathematics at the University of Wyoming, has been appointed to an assistant professorship.

Associate Professor J. W. Calhoun, of the University of Texas, has been promoted to a full professorship of applied mathematics.

Mr. W. M. Whyburn, of the University of Texas, has been appointed professor of mathematics at South Park College.

The following have been appointed instructors in mathematics:
Carnegie Institute of Technology, Mr. J. H. Simester;
Colgate University, Messrs. J. C. Polley and G. O. Hohl;
Dalhousie University, Dr. F. H. Murray;
University of Georgia, Messrs. F. Cumming and E. M. Everett;
Georgia School of Technology, Messrs. T. P. Branch, J. G. Griffin, J. G. Evans, W. W. Purks, J. W. Taylor;
Hamline University, Mr. E. L. Michelson;
Iowa State College, Mr. B. A. Rogers;
University of Kentucky, Messrs. T. Andrew and J. N. Nixon;
Purdue University, Mr. C. E. Barr;
Rockford College, Miss Martha P. McGavock;
University of South Carolina, Mr. S. T. Sparkman;
Texas Agricultural and Mechanical College, Mr. H. A. Robinson.

Lady Shaw, wife of Sir Napier Shaw, died September 22, 1923. She was at one time lecturer in mathematics at Newnham College, Cambridge.

Professor Malcolm McNeill, who had been head of the department of mathematics at Lake Forest College for thirty-five years, died suddenly October 5, 1923.

Professor H. F. Stecker, of Pennsylvania State College, died October 30, 1923, at the age of fifty-six years. Professor Stecker had been a member of the American Mathematical Society since 1894.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARCHIMEDES. Ueber Paraboloide, Hyperboloide und Ellipsoide. (Ostwald's Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1923. 73 pp.

BJERRUM (N.). See BRODÉN (T.).

BORCHARDT (L.). Gegen die Zahlenmystik an der großen Pyramide bei Gise. Berlin, Behrend, 1923. 40 pp.

BOUVIER (R.). La pensée d'Ernest Mach. Essai de biographie intellectuelle et de critique. Paris, chez l'auteur, 1922. 370 pp.

BRODÉN (T.), BJERRUM (N.) och STRÖMGREN (E.). Matematiken och de exakten naturvetenskaperne under det nittonde århundradet. Stockholm, Norstedt, 1922. 248 pp.

DUMOULIN (A.). See MANSION (P.).

DURELL (C. V.). See FAWDRY (R. C.).

FAWDRY (R. C.) and DURELL (C. V.). Calculus for schools. London, E. Arnold, 1923. 8 + 300 + 20 pp.

VAN GEER (P.). See WOLFF (J.).

LORENZ (H.). Einführung in die Elemente der höheren Mathematik und Mechanik. München, Oldenbourg, 1923. 176 pp.

MANSION (P.). L'essence de la trigonométrie. Avec le portrait de l'auteur et une notice biographique par A. Dumoulin. Paris, Gauthier-Villars, 1922.

NATUCCI (A.). Il concetto di numero e le sue estensioni. Torino, Fratelli Bocca, 1923. 8 + 474 pp.

ROSE (W. N.). Mathematics for engineers. 2d edition. Part 2. London, Chapman and Hall, 1923. 14 + 423 pp.

RUTLEDGE (G.). Fundamental topics in the differential and integral calculus. Boston, Ginn, 1923. 8 + 252 pp.

SMITH (D. E.). Mathematics. (In the series Our Debt to Greece and Rome.) Boston, Marshall Jones Company, 1923. 10 + 175 pp.

STRÖMGREN (E.). See BRODÉN (T.).

TRIPIER (H.). Les fonctions circulaires et les fonctions hyperboliques étudiées parallèlement en partant de la définition géométrique. Paris, Librairie Vuibert, 1923. 4 + 58 pp.

WEIDEMANN (-.). Zauberquadrate und andere magische Zahlenfiguren der Ebene und des Raumes. Leipzig, Oskar Leiner, 1922. 83 pp.

WITTING (A.). Abgekürzte Rechnung nebst Einführung in die Rechnung mit Logarithmen. (Mathematisch-Physikalische Bibliothek, Band 48.) Leipzig, Teubner, 1922.

WOLFF (J.). Inleiding tot de analytische meetkunde van it platte vlak. Uitgegeven in de plaats van Dr. P. van Geer's Leerboek der analytische meetkunde, eerste deel. Groningen, Noordhoff, 1922.

PART II. APPLIED MATHEMATICS

ABADIE-DUTEMPS (—.). Notice sur l'hypsonome. Instrument qui donne les plans cotés sans calculs, les hauteurs des points ou la situation de points dont la cote est donnée. Paris, Gauthier-Villars, 1923. 64 pp.

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique à l'usage des élèves des classes de mathématiques A et B. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1923. 9 + 416 pp.

AUERBACH (F.). Entwicklungsgeschichte der modernen Physik, zugleich eine Uebersicht ihrer Tatsachen, Gesetze und Theorien. Berlin, Springer, 1923. 8 + 344 pp.

BABINI (J.). Termodinámica gráfica. Buenos Aires, Revista del Centro de Estudiantes de Ingenieria, 1923. 56 pp.

BATEMAN (H.). See SWANN (W. F. G.).

BLANC (A.). Le chef mécanicien-électricien. Tome I: Mathématiques pratiques. Paris, Librairie Desforges, 1923. 16 + 319 pp.

BOHR (N.). Les spectres et la structure de l'atome. Traduit par A. Corvisy. Paris, Hermann, 1923. 8vo. 152 pp.

BOREL (E.). See CAMPBELL (N. R.).

BOUASSE (H.) et CARRIÈRE (Z.). Diffraction. Paris, Librairie Delagrave, 1923. 22 + 480 pp.

BÜCHLER (R.). Ueber die Einsteinsche Relativitätstheorie. Aachen, 1922. 4 pp.

CAMPBELL (N. R.). Les principes de la physique. Préface de M. E. Borel. Paris, Alcan, 1923. 19 + 200 pp.

CARRIÈRE (Z.). See BOUASSE (H.).

CERMAK (P.). Die Röntgenstrahlen. Leipzig, Barth, 1923. 130 pp.

CHAPPUIS (J.). See APPELL (P.).

COLEMAN (G. S.). Calculations in heating and ventilation. London, Longmans, 1923. 19 + 255 pp.

COLLINS (A. F.). The book of the microscope. New York, Appleton, 1923. 15 + 245 pp.

CORPS (—.). La simultanéité générale et le temps universelle. Paris, Gauthier-Villars, 1923.

CORVISY (A.). See BOHR (N.).

CROFT (T.). Steam-turbine principles and practice. London, McGraw-Hill, 1923. 11 + 347 pp.

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EASON (A. B.). The prevention of vibration and noise. (Oxford Technical Series.) London, H. Frowde and Hodder and Stoughton, 1923. 12 + 163 pp.

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GLEICHEN (A.). Die Theorie der modernen optischen Instrumente. 2te, neubearbeitete Auflage. Stuttgart. Enke, 1923.

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TER HEERDT (J.) Overzicht van de theorie en de toepassingen van gassen waarin de onderlinge botsingen der moleculen kunnen verwaardloosed woorden. Utrecht, N. V. Dekker, 1923. 7 + 324 pp.

HILLERS (W.). See GRIMSEHL (E.).

HOBART (H. M.). Electric motors, their theory and construction. 3d edition, revised throughout. Volumes 1–2. London, Pitman, 1923. 16 + 412 + 15 + 384 pp.

HUBER (M. T.). The theory of rectangular anisotropic plates, with particular reference to reinforced concrete plates. (In Polish.) Lemberg, 1921.

KELLEY (T. L.). Statistical method. New York, Macmillan, 1923. 11 + 300 pp.

KENNARD (E. H.). See SWANN (W. F. G.).

KOSACK (E.). Elektrische Starkstromanlagen. 6te, durchgesehene und ergänzte Auflage. Berlin, Springer, 1923.

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LORAIN (P.). L'hélice propulsive. Etude théorique et expérimentale du propulseur marin. Applications pratiques. Paris, Dunod, 1923. 196 pp.

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MACCALL (W. T.). Alternate current electrical engineering. London, University Tutorial Press, 1923. 8 + 493 pp.

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Poeschl (T.). Lehrbuch der technischen Mechanik. Berlin, Springer, 1923. 6 + 264 pp.

Premier Congrès International des Sociétés Astronomiques. Rapport général. Toulouse, Edouard Privat, 1923.

Prescott (J.). Mechanics of particles and rigid bodies. 2d edition. New York and London, Longmans, 1923. 6 + 535 pp.

Rice (J.). Relativity. A systematic treatment of Einstein's theory. New York and London, Longmans, 1923. 16 + 397 pp. $6.00

Schott (S.). Statistik. 3te Auflage. Leipzig, Teubner, 1923. 128 pp.

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